Reachability analysis: Integer Timed Automaton

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This course so far

- A modeling framework
  - Discrete and continuous dynamics
  - Compositional (modular) modeling
- General proof techniques for proving invariants

Next

- Focus on specific classes of Hybrid Automata for which safety properties (invariants) can be verified completely automatically
  - Alur-Dill's Timed Automata[1] (Today)
  - Rectangular initializaed hybrid automata
  - Linear hybrid automata
  - …
- Later we will look at other types of properties like stability, liveness, etc.
- We will introduce notions of abstractions and invariance are still going to be important

[1] Rajeev Alur et al. The Algorithmic Analysis ofHybrid Systems. Theoretical Computer Science, colume 138, pages 3-34, 1995.

Today

- Algorithmic analysis of (Alur-Dill's) Timed Automata[1]
  – A restricted class of what we call hybrid automata in this course with only clock variables

[1] Rajeev Alur and David L. Dill. A theory of timed automata. Theoretical Computer Science, 126:183-235, 1994.

Clocks and Clock Constraints

- A clock variable x is a continuous (analog) variable of type real such that along any trajectory τ of x, for all t ∈ $\tau.dom, (\tau \downarrow x)(t) = t$.

- For a set X of clock variables, the set Φ(X) of integral clock constraints are expressions defined by the syntax:
  g ::= x $\leq q \mid x \geq q \mid \neg\, g \mid g_1 \wedge\ g_2$
  where $x \in X$ and $q \in \mathbb{Z}$

- Examples: x = 10; x ∈ [2, 5); true are valid clock constraints
- What do clock constraints look like?

- Semantics of clock constraints $[g]$

Integral Timed Automata

- **Definition.** A integral timed automaton is a HIOA $\mathcal{A} = \langle V, \Theta, A, \mathcal{D}, \mathcal{T} \rangle$ where
  - V = X ∪ $\{l\}$, where X is a set of n clocks and l is a discrete state variable of finite type L
  - A is a finite set
  - $\mathcal{D}$ is a set of transitions such that
    - The guards are described by clock constrainGS $\Phi(X)$
    - $\langle x, l \rangle - a \rightarrow \langle x', l' \rangle$ implies either $x' = x$ or $x = 0$
  - $\mathcal{T}$ set of clock trajectories for the clock variables in X

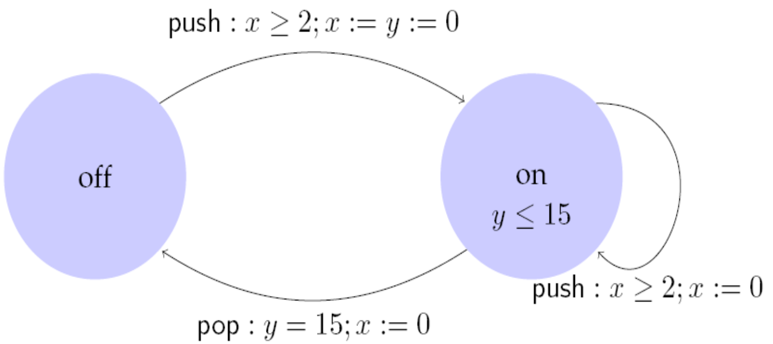
Example: Light switch

Math Formulation
automaton Switch
 variables
 internal x, y:Real := 0, loc: {on,off} := off

 transitions
 internal push
  pre x $\geq$ 2
  eff if loc = on **then** x := 0
   else x,y := 0; loc := off
 internal pop
  pre y = 15 $\bigwedge$ loc = off
  eff x := 0

 trajectories
  invariant loc = off => y $\leq$ 15
  evolve d(x) = 1; d(y) = 1

Description
Switch can be turned on whenever at least 2 time units have elapsed since the last turn on. Switches off automatically 15 time units after the last on.



Control State (Location) Reachability Problem

- Given an ITA $\mathcal{A}$, check if a particular (discrete) control state is reachable from the initial states
- Why is control state reachability (CSR) good enough?
- This problem is decidable [Alur Dill]
- Key idea:
  - Construct a finite automaton that is a time-abstract ***bisimilar*** to the ITA (behaves identically with respect to control state reachability)
  - Check reachability of FSM

An equivalence relation with a finite quotient

- Under what conditions do two states x_1 and x_2 of the automaton $\mathcal{A}$ behave identically with respect to control state reachability (CSR)?
  - When do they satisfy the same set of clock constraints?
  - When would they continue to satisfy the same set of clock constraints?

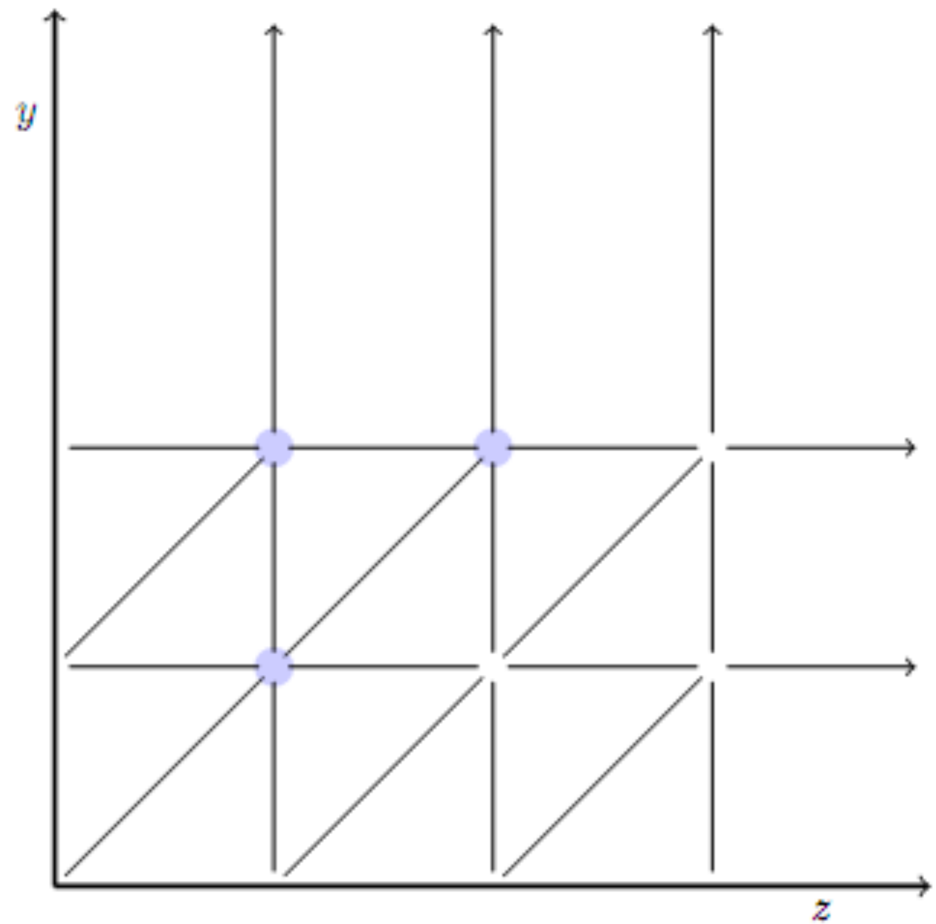
- $x_1.loc = x_2.loc$ and
- x_1 and x_2 satisfy the same set of clock constraints
  - For each clock y int($x_1.y$) = int($x_2.y$) or int$(x_1.y) \geq c_{\mathcal{A}y}$ and int$(x_2.y) \geq c_{\mathcal{A}y}$. ($c_{\mathcal{A}y}$ **is the maxium clock guard** of y)
  - For each clock y with $x_1.y \leq c_{\mathcal{A}y}$, frac$(x_1.y) = 0$ iff frac$(x_2.y) = 0$
  - For any two clocks y and z with $x_1.y \leq c_{\mathcal{A}y}$ and $x_1.z \leq c_{\mathcal{A}z}$, frac($x_1.y$) $\leq$ frac($x_1.z$) iff frac($x_2.y$) $\leq$ frac($x_2.z$)
- Lemma. This is a **equivalence relation** on *val(V)* the states of $\mathcal{A}$

- The partition of *val(V)* induced by this relation is are called **clock regions**

What do the clock regions look like?

Example of
Two Clocks

X = {y,z}
$c_{\mathcal{A}y}$ = 2
$c_{\mathcal{A}z}$ = 3



Complexity

- **Lemma**. The number of clock regions is bounded by $|X|!\, 2^{|X|} \prod_{z \in X} (2c_{\mathcal{A}z} + 2)$.

Region automaton $\mathrm{R}(\mathcal{A})$

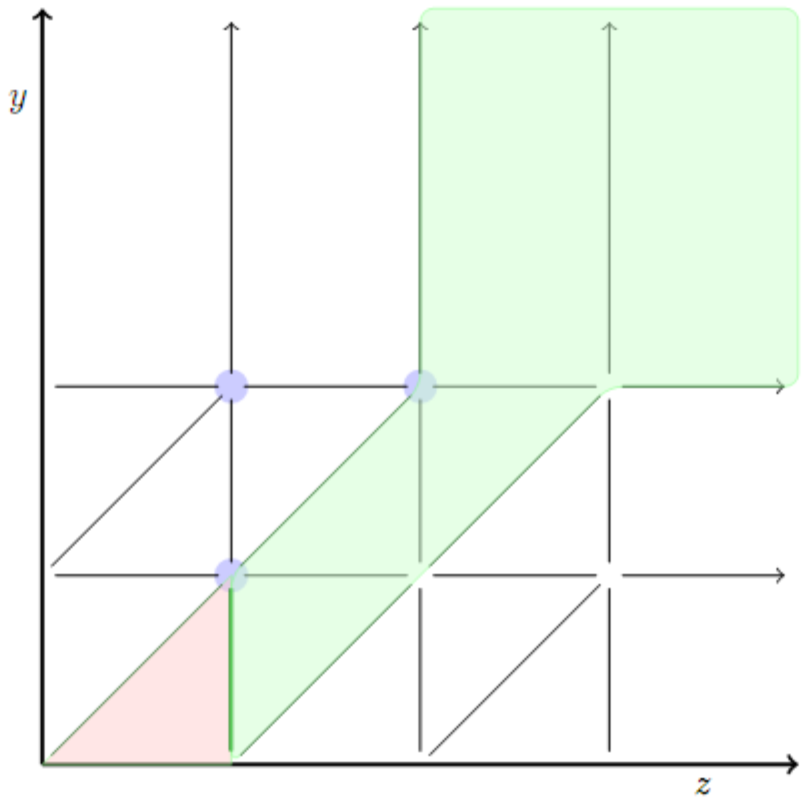
Given an ITA $\mathcal{A} = \langle V, \Theta, \mathcal{D}, \mathcal{T} \rangle$, we construct the corresponding **Region Automaton** $\mathrm{R}(\mathcal{A}) = \langle Q_R, \Theta_R, D_R \rangle$ such that (i) $\mathrm{R}(\mathcal{A})$ visits the same set of locations (but does not have timing information) and (ii) $\mathrm{R}(\mathcal{A})$ is finite state machine.

- ITA (clock constants) defines a set of clock regions, say $\mathrm{C}_{\mathcal{A}}$. The set of states $Q_R = C_{\mathcal{A}} \times L$
- $Q_0 \subseteq Q$is the set of states contain initial set Θ of $\mathcal{A}$
- D: We add the transitions between Q (regions)
  - **Time successors**: Consider two clock regions γ and γ', we say that γ' is a time successor of γ if there exits a trajectory of ITA starting from γ that ends in γ'
  - **Discrete transitions**: Same as the ITA

Theorem. A location of ITA $\mathcal{A}$ is reachable iff it is also reachable in $\mathrm{R}(\mathcal{A})$.

(we say that $\mathrm{R}(\mathcal{A})$ is *time abstract bisimilar* to $\mathcal{A}$)

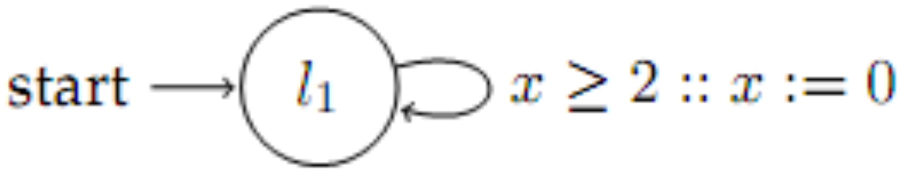
Time successors



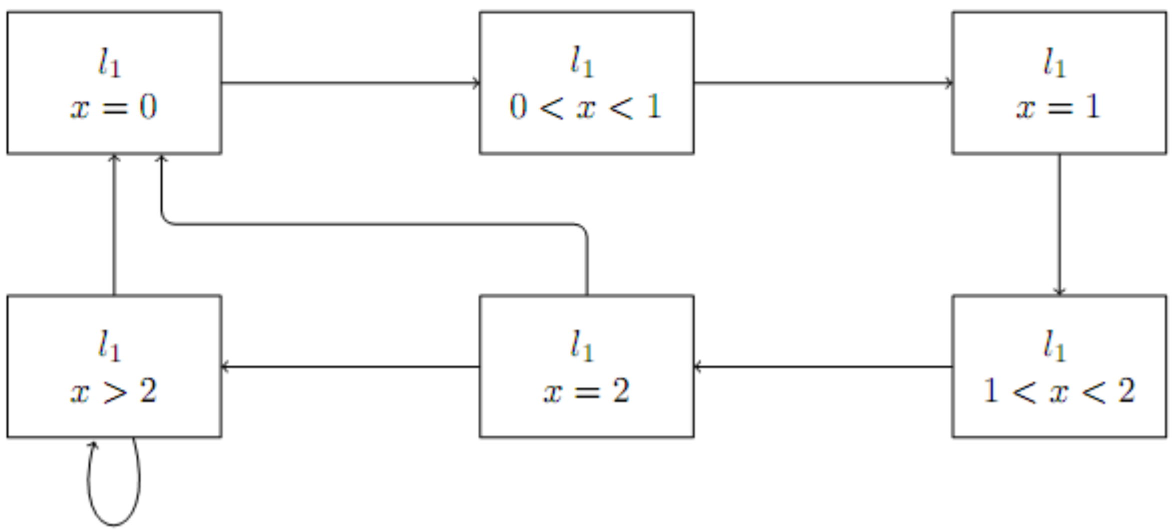
The clock regions in blue are time successors of the clock region in red.

Example 1: Region Automata

ITA

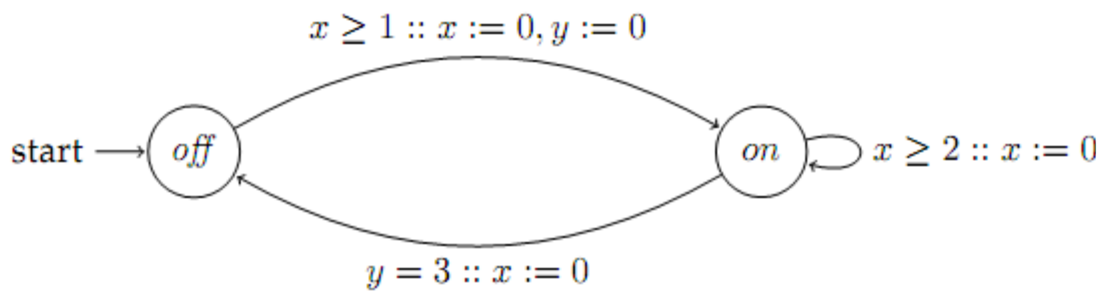


Corresponding FA

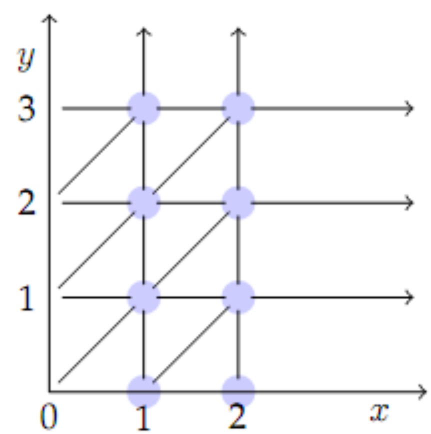


Example 2

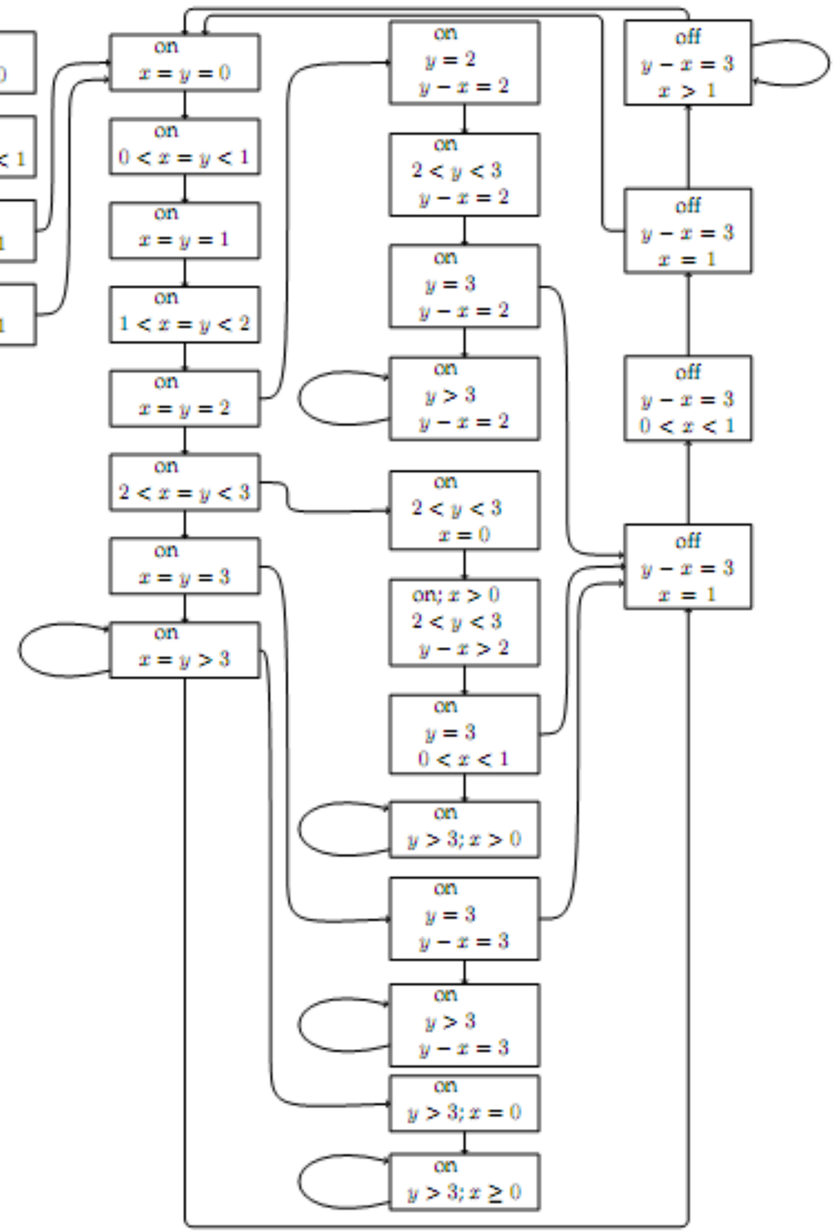
ITA



Clock Regions



Corresponding FA



$|X|!\, 2^{|X|} \prod_{z \in X} (2 c_{\mathcal{A}z} + 2)$

Drastically increasing with the number of clocks

Special Classes of Hybrid Automata

- Timed Automata ←
- Rational time automata
- Multirate automata
- Rectangular Initialized HA
- Rectangular HA
- Linear HA
- Nonlinear HA

Clocks and **Rational** Clock Constraints

- A **clock variable** x is a continuous (analog) variable of type real such that along any trajectory τ of x, for all t ∈ $\tau.dom, (\tau \downarrow x)(t) = t$.

- For a set X of clock variables, the set Φ(X) of ***rational* clock constraints** are expressions defined by the syntax:
  g ::= x $\leq q \mid x \geq q \mid \neg\, g \mid g_1 \wedge \; g_2$
  where $x \in X \; and \; q \in \; \mathbb{Q}$

- Examples: x = 10.125; x ∈ [2.99, 5); true are valid rational clock constraints

- Semantics of clock constraints $[g]$

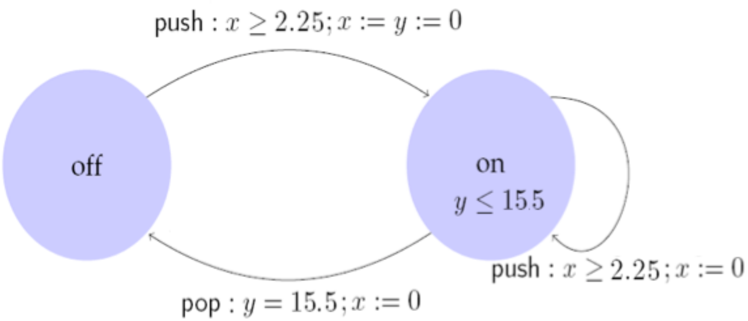
Step 1. Rational Timed Automata

- **Definition.** A *rational timed automaton* is a HA $\boldsymbol{\mathcal{A}}$ = $\langle V, \Theta, A, \mathcal{D}, \mathcal{T} \rangle$ where
  - V = X ∪ $\{loc\}$, where X is a set of n clocks and l is a discrete state variable of finite type Ł
  - A is a finite set
  - $\mathcal{D}$ is a set of transitions such that
    - The guards are described by rational clock constrings $\Phi(X)$
    - $\langle x, l \rangle - a \rightarrow \langle x', l' \rangle$ implies either $x' = x$ or $x = 0$
  - $\mathcal{T}$ set of clock trajectories for the clock variables in X

Example: Rational Light switch

Switch can be turned on whenever at least 2.25 time units have elapsed since the last turn off or on. Switches off automatically 15.5 time units after the last on.

automaton Switch
  internal push; pop
    variables
      internal x, y:Real := 0, loc:{on,off} := off
    transitions
      push
        pre x >=2.25
        eff if loc = on **then** y := 0 **fi;** x := 0; loc := off
      pop
        pre y = 15.5 ∧ loc = off
        eff x := 0
    trajectories
      invariant loc = on ∨ loc = off
      stop when y = 15.5 ∧ loc = off
      evolve d(x) = 1; d(y) = 1



Control State (Location) Reachability Problem

- Given an RTA, check if a particular location is reachable from the initial states
- Is problem decidable?
- Yes
- Key idea:
  – Construct a ITA that is time-abstract bisimilar to the given RTA
  – Check CSR for ITA

Construction of ITA from RTA

- Multiply all rational constants by a factor q that make them integral
- Make d(x) = q for all the clocks

- RTA Switch is bisimilar to ITA Iswitch

- Simulation relation R is given by
- (**u**,**s**) $\in R$ iff **u.**x = 4 **s.**x and **u.**y = 4 **s.**y

automaton ISwitch
internal push; pop
variables
  internal x, y:Real := 0, loc:{on,off} := off
transitions
  push
    pre x >= 9
    eff if loc = on **then** y := 0 **fi;** x := 0; loc := off
  pop
    pre y = 62 ∧ loc = off
    eff x := 0
trajectories
  invariant loc = on ∨ loc = off
  stop when y = 62 ∧ loc = off
  evolve d(x) = 4; d(y) = 4

Step 2. Multi-Rate Automaton

- **Definition.** A multirate automaton is $\boldsymbol{\mathcal{A}} = \langle V, Q, \Theta, A, \mathcal{D}, \mathcal{T} \rangle$ where
  - V = X ∪ $\{loc\}$, where X is a set of n continuous variables and loc is a discrete state variable of finite type Ł
  - A is a finite set of actions
  - $\mathcal{D}$ is a set of transitions such that
    - The guards are described by rational clock constrings $\Phi(X)$
    - $\langle x, l \rangle - a \rightarrow \langle x', l' \rangle$ implies either $x' = c \; or \; x' = x$
  - $\mathcal{T}$ set of trajectories such that

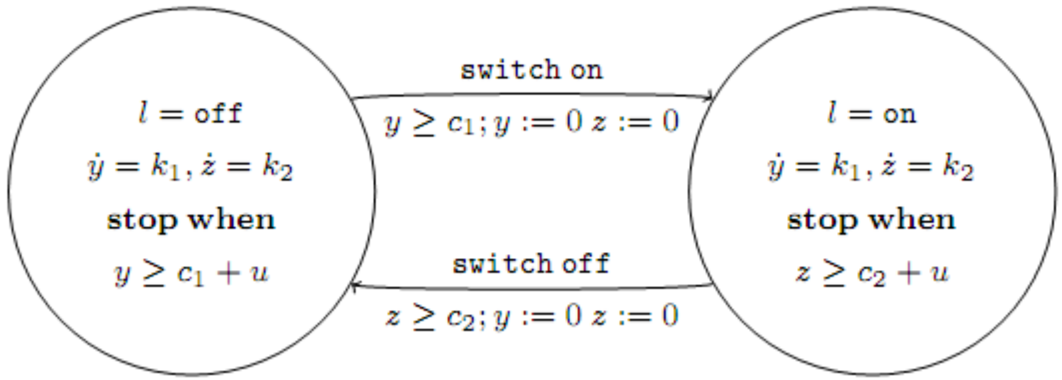
    for each variable $x \in X \; \exists k \; such \; that \; \tau \in \mathcal{T}, t \in \tau.dom$

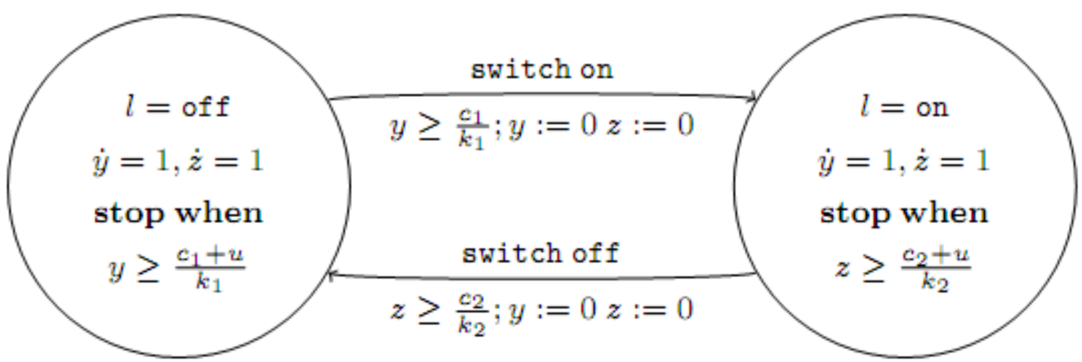
$$\tau(t).x = \tau(0).x + k\,t$$

Control State (Location) Reachability Problem

- Given an MRA, check if a particular location is reachable from the initial states
- Is problem is decidable? Yes
- Key idea:
  – Construct a RTA that is bisimilar to the given MRA

Example: Multi-rate to rational TA





Step 3. Rectangular HA

Definition. An rectangular hybrid automaton (RHA) is a HA $\boldsymbol{\mathcal{A}} = \langle V, A, \mathcal{T}, \mathcal{D} \rangle$ where

- V = X ∪ $\{loc\}$, where X is a set of n continuous variables and loc is a discrete state variable of finite type Ł
- A is a finite set
- $\mathcal{T} = \cup_\ell \mathcal{T}_\ell$ set of trajectories for X
  - For each $\tau \in \mathcal{T}_\ell, x \in X$ either (i) $d(x) = k_\ell$ or (ii) $d(x) \in [k_{\ell 1}, k_{\ell 2}]$
  - Equivalently, (i) $\tau(t) \lceil x = \tau(0) \lceil x + k_\ell t$

    (ii) $\tau(0) \lceil x + k_{\ell 1} t \leq \tau(t) \lceil x \leq \tau(0) \lceil x + k_{\ell 2} t$
- $\mathcal{D}$ is a set of transitions such that
  - Guards are described by rational clock constraings
  - $\langle x, l \rangle \rightarrow_a \langle x', l' \rangle$ implies $x' = x \; or x' \in [c_1, c_2]$

CSR Decidable for RHA?

- Given an RHA, check if a particular location is reachable from the initial states?
- Is this problem decidable? No
  - [Henz95] Thomas Henzinger, Peter Kopke, Anuj Puri, and Pravin Varaiya. What's Decidable About Hybrid Automata?. Journal of Computer and System Sciences, pages 373–382. ACM Press, 1995.
  - CSR for RHA reduction to Halting problem for 2 counter machines
  - Halting problem for 2CM known to be undecidable
  - Reduction in next lecture

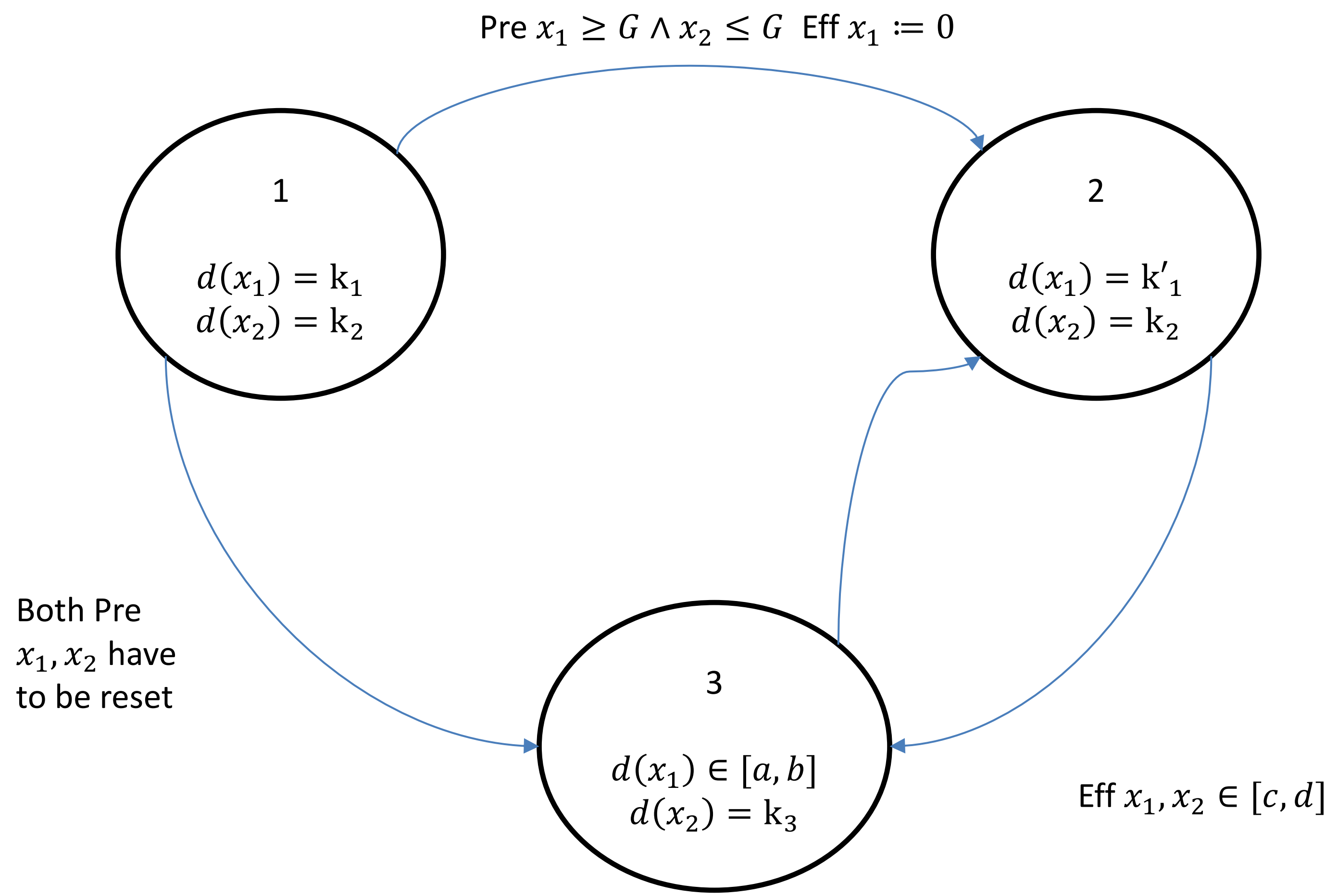
Step 4. Initialized Rectangular HA

Definition. An initialized rectangular hybrid automaton (IRHA) is a RHA $\boldsymbol{\mathcal{A}}$ where

- V = X ∪ $\{loc\}$, where X is a set of n continuous variables and $\{loc\}$ is a discrete state variable of finite type Ł
- A is a finite set
- $\mathcal{T} = \cup_{\ell} \mathcal{T}_{\ell}$ set of trajectories for X
  - For each $\tau \in \mathcal{T}_{\ell}, x \in X$ either (i) $d(x) = k_{\ell}$ or (ii) $d(x) \in [k_{\ell 1}, k_{\ell 2}]$
  - Equivalently, (i) $\tau(t)\lceil x = \tau(0)\lceil x + k_{\ell} t$

    (ii) $\tau(0)\lceil x + k_{\ell 1} t \leq \tau(t)\lceil x \leq \tau(0)\lceil x + k_{\ell 2} t$
- $\mathcal{D}$ is a set of transitions such that
  - Guards are described by rational clock constrings
  - $\langle x, l \rangle \rightarrow_a \langle x', l' \rangle$ implies if dynamics changes from ℓ to ℓ' then $x' \in [c_1, c_2]$, otherwise $x' = x$

Example: Rectangular Initialized HA

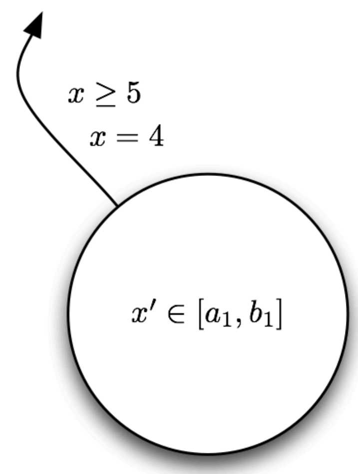


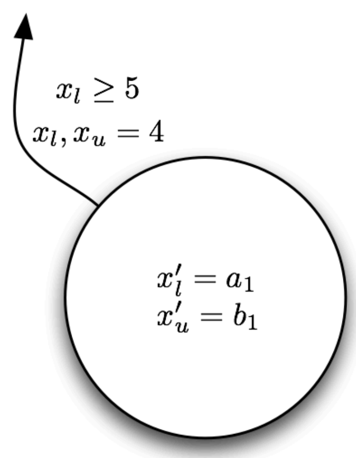
CSR Decidable for IRHA?

- Given an IRHA, check if a particular location is reachable from the initial states
- Is this problem decidable? Yes
- Key idea:
  - Construct a 2n-dimensional **initialized m**ulti-rate automaton that is bisimilar to the given IRHA
  - Construct a ITA that is bisimilar to the Singular TA

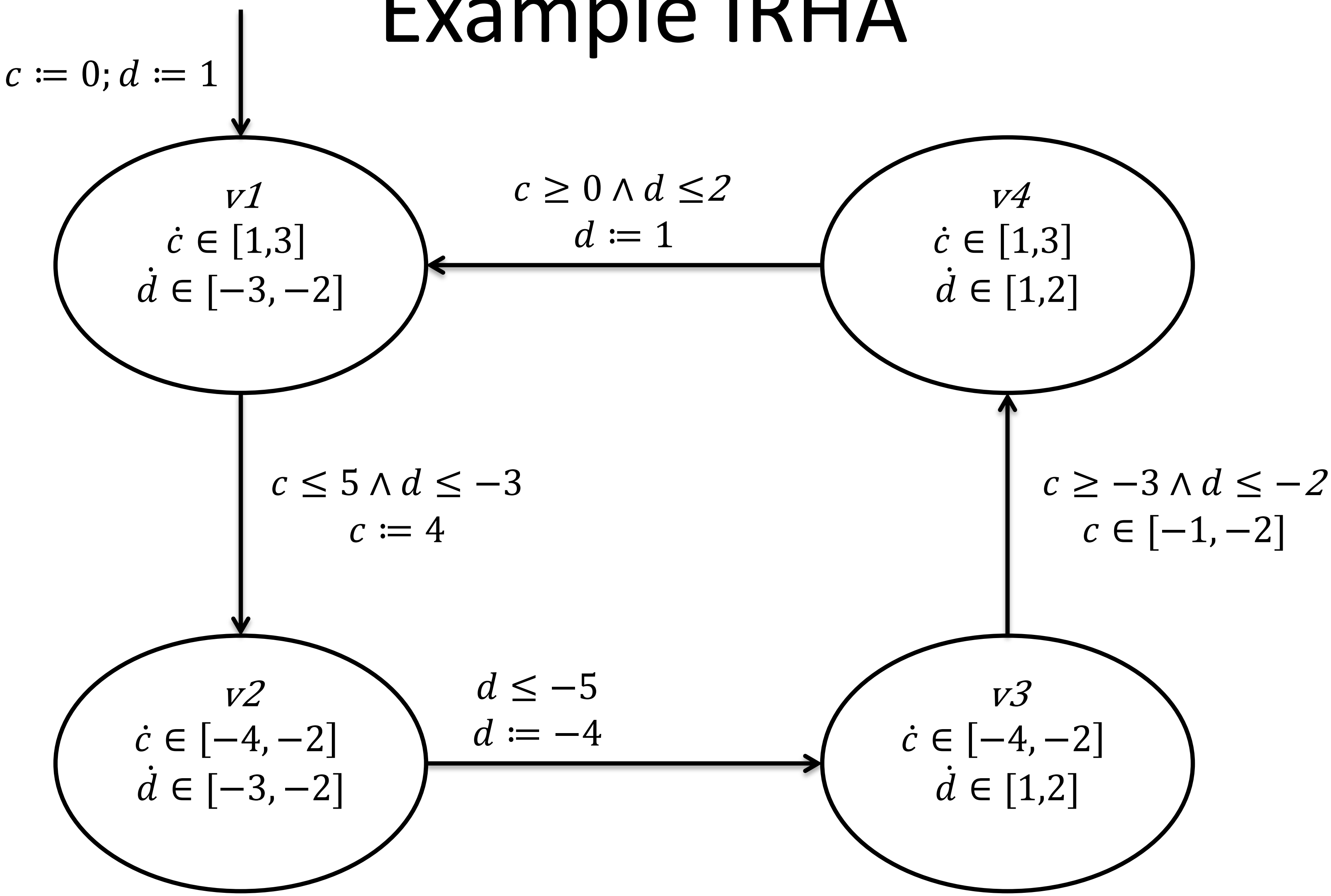
Split every variable into two variables---tracking the upper and lower bounds

IRHA	MRA
x	x_ℓ ; x_u
Evolve: $d(x) \in [a_1, b_1]$	Evolve: $d(x_\ell) = a_1; d(x_u) = b_1$
Eff: $x' \in [a_1, b_1]$	Eff: $x_\ell = a_1; x_u = b_1$
$x' = c$	$x_\ell = x_u = c$
Guard: $x \geq 5$	$x_l \geq 5$
	$x_l < 5 \wedge x_u \geq 5$ Eff $x_l = 5$

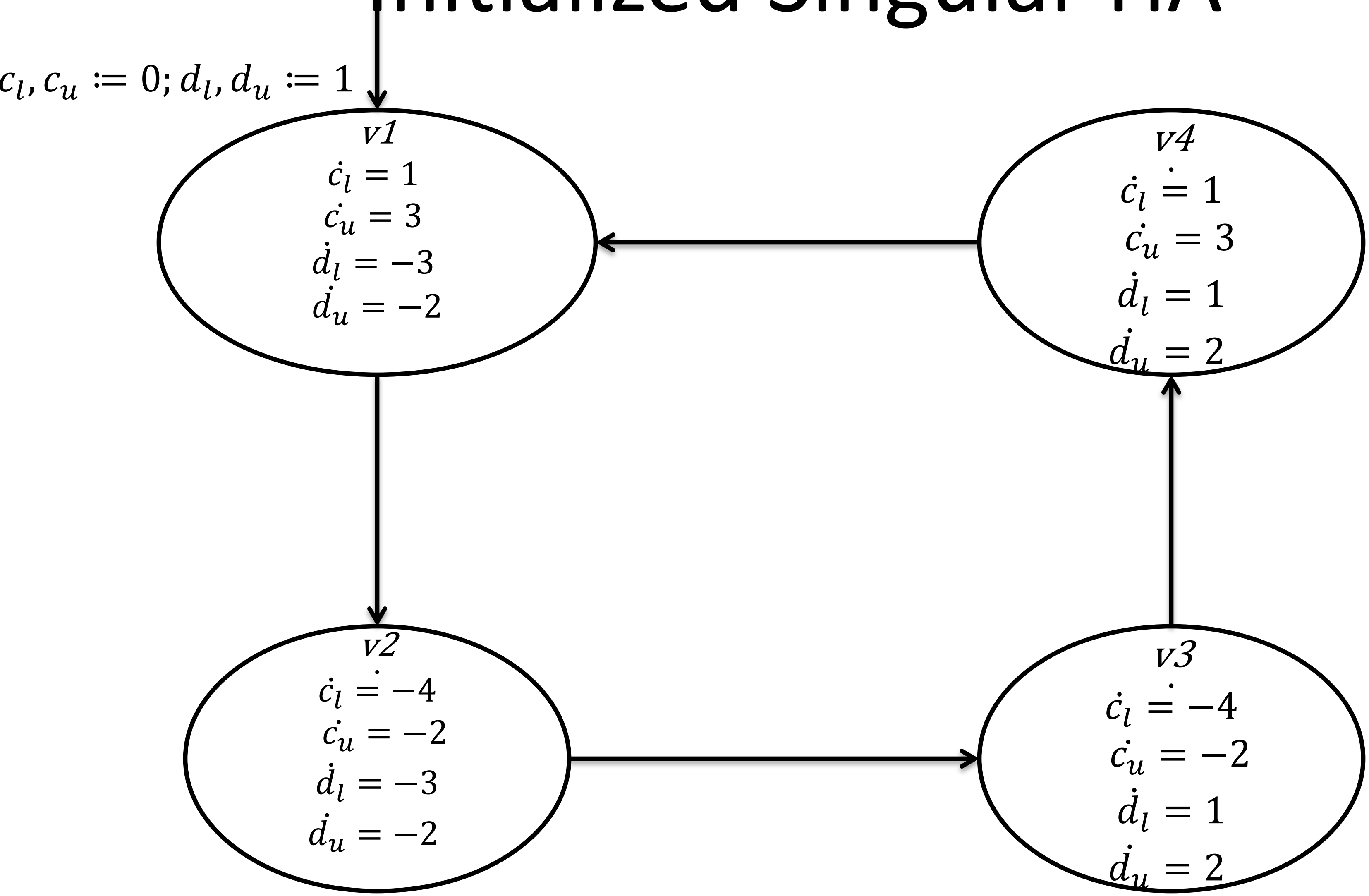




Example IRHA



Initialized Singular HA



Transitions

v1

$\dot{c}_l = 1$

$\dot{c}_u = 3$

$\dot{d}_l = -3$

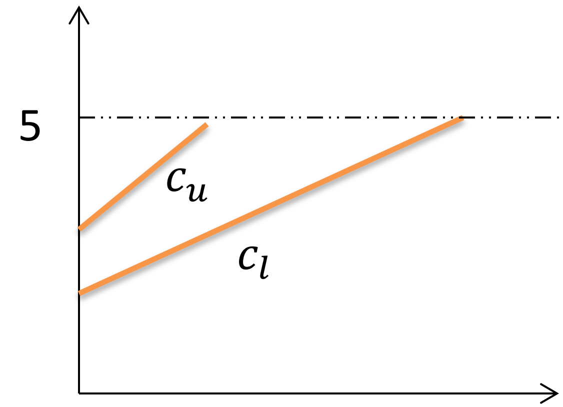
$\dot{d}_u = -2$

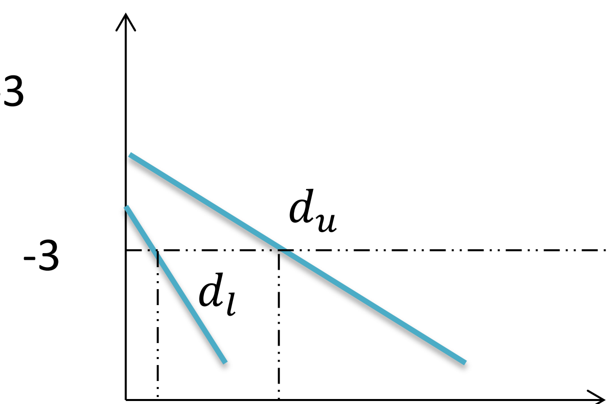
$c_l \leq 5$

$c_l, c_u \coloneqq 4$

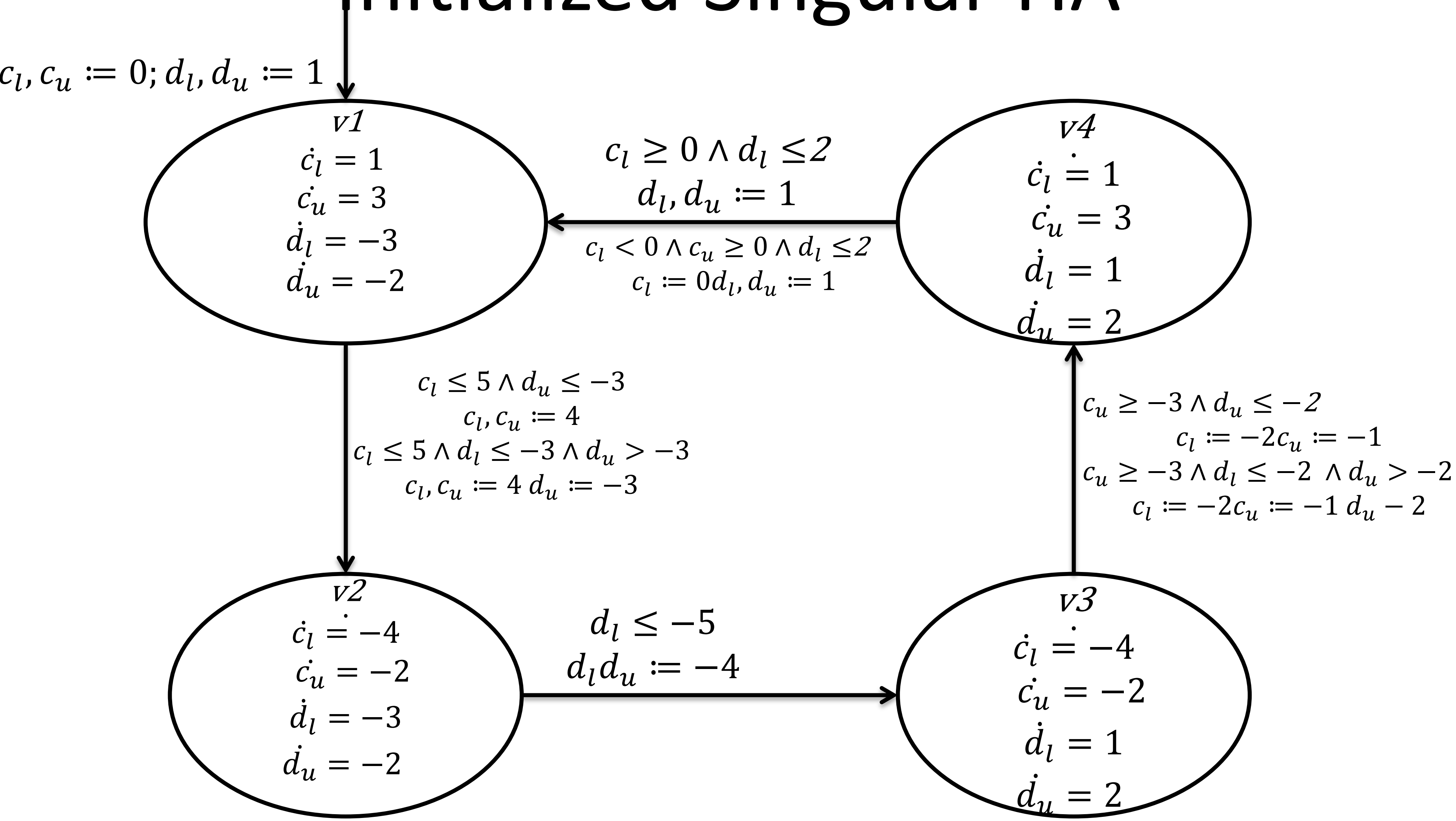
$d_u \leq -3$ *no reset*

$d_u > -3 \wedge d_l \leq -3$ $d_u \coloneqq$-3





Initialized Singular HA



Can this be further generalized ?

- For initialized Rectangular HA, control state reachability is decidable
  - Can we drop the initialization restriction?
    - No, problem becomes undecidable (next time)
  - Can we drop the rectangular restriction?
    - No, problem becomes undecidable

Verification in tools

Algorithm: BasicReach
Input: $\mathbf{A} = \langle V, \Theta, A, \mathbf{D}, \mathbf{T} \rangle$, $d > 0$
  $Rt, Reach: val(V)$
  $Rt := \Theta;$
  $Reach := \emptyset;$
  While ($Rt \not\subseteq$ Reach)
    $Reach := Reach \cup Rt;$
    $Rt := Rt \cup Post_{\mathbf{D}}(Rt);$
    $Rt := Post_{\mathbf{T}(d)}(Rt);$
Output: $Reach$

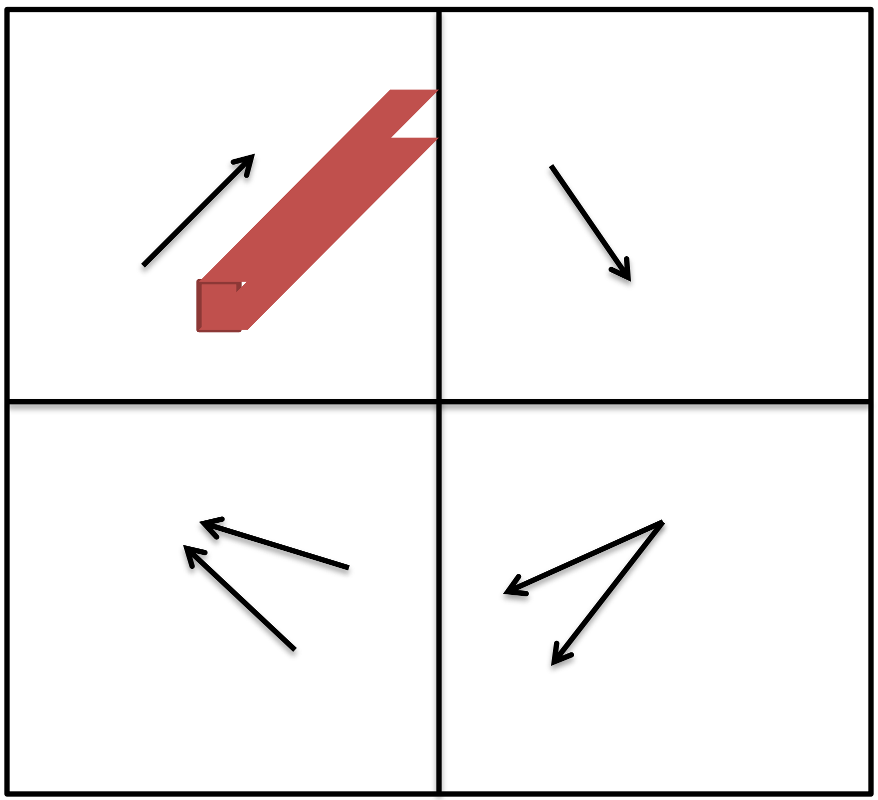
Algorithm: $\mathrm{Post}_{\mathrm{D}}$
\\\\ **computes post of all transitions**
Input: $A, \mathbf{D}, S_{in}$
  $S_{out} = \emptyset$
  For each $a \in A$
    For each $\langle g_1, g_2 \rangle \in S_{in}$
      If $[\![g_1, g_2]\!] \cap [\![g_{ga1}, g_{ga2}]\!] \neq \emptyset$
        $S_{out} := S_{out} \cup \langle g_{ra1}, g_{gra2} \rangle$
Output: S_{out}

Algorithm: $\mathrm{Post}_{\mathbf{T(d)}}$
\\\\ **computes post of all trajectories**
Input: $A, \mathbf{T}, S_{in}, d$
  $S_{out} = \emptyset$
  For each $\ell \in L$
    For each $\langle g_1, g_2 \rangle \in S_{in}$
      $P := \cup_{t \leq d} [\![g_1, g_2]\!] \oplus [\![t g_{\ell 1}, t g_{\ell 2}]\!]$
      $S_{out} := S_{out} \cup Approx(P)$
Output: S_{out}

Data structures make reachability go around

- Hyperrectangles
  - $[[\mathrm{g}_1; \mathrm{g}_2]] = \{x \in R^n \mid \ \left||\mathrm{x} - \mathrm{g}_1|\right|_\infty \le \left||\mathrm{g}_2 - \mathrm{g}_1|\right|_\infty\} = \Pi_i[g_{1i}, g_{2i}]$
- Polyhedra
- Zonotopes [Girard 2005]
- Ellipsoids [Kurzhanskiy 2001]
- Support functions [Guernic et al. 2009]
- Generalized star set [Duggirala and Viswanathan 2018]

Reachability Computation with polyhedra



Portion of Navigation benchmark

- A set of states is represented by disjunction of linear inequalities
  - $(loc = l_1 \wedge A_1 x \leq b_1) \vee$
    $(loc = l_2 \wedge A_2 x \leq b_2) \vee$
    $\ldots$
- Post(,) computation performed symbolically using quantifier elimination

$$x' = k \rightarrow Post([a_1, a_2]) = \exists t\ [a_1 + kt, a_2 + kt] = [a_1, \infty]$$

the state is reachable if there exists a time when we reach it.

Summary

- ITA: (very) Restricted class of hybrid automata
  - Clocks, integer constraints
  - No clock comparison, linear
- Control state reachability with Alur-Dill's algorithm (region automaton construction)
- Rational coefficients
- Multirate Automata
- Initialized Rectangular Hybrid Automata
- HyTech, PHAVer use polyhedral reachability computations