Modeling Physics

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Plan

- Dynamical system models
  - notions of solutions
  - Linear dynamical systems
  - Connection to automata
  - Stability
  - Lyapunov method

Map of CPS models

Discrete transition systems, automata

Dynamical systems
Differential inclusions

Hybrid systems

Markov chains

Probabilistic automata, Markov decision processes (MDP)

Continuous time, continuous state MDPs

Stochastic Hybrid systems

All this was in the two plague years 1665 and 1666, for in those days I was in my prime of age for invention, and minded mathematics and philosophy more than at any time since.

---Isaac Newton

From: Wilczek, Frank. A Beautiful Question: Finding Nature's Deep Design (p. 87).

Introduction to dynamical systems

Behaviors of physical processes are described in terms of instantaneous laws

Common notation: $\frac{dx(t)}{dt} = f(x(t), u(t), t) \ - (1)$,
where time $t \in \mathbb{R}$; state $x(t) \in \mathbb{R}^n$; $input \ \ u(t) \in \mathbb{R}^m$; $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$

Example. $\frac{dx(t)}{dt} = v(t)$; $\frac{dv(t)}{dt} = -g$

Initial value problem: Given system (1) and initial state $x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}$, and input u: $\mathbb{R} \to \mathbb{R}^m$, find a state trajectory or *solution* of (1).

Notions of solution

What is a solution? Many different notions.

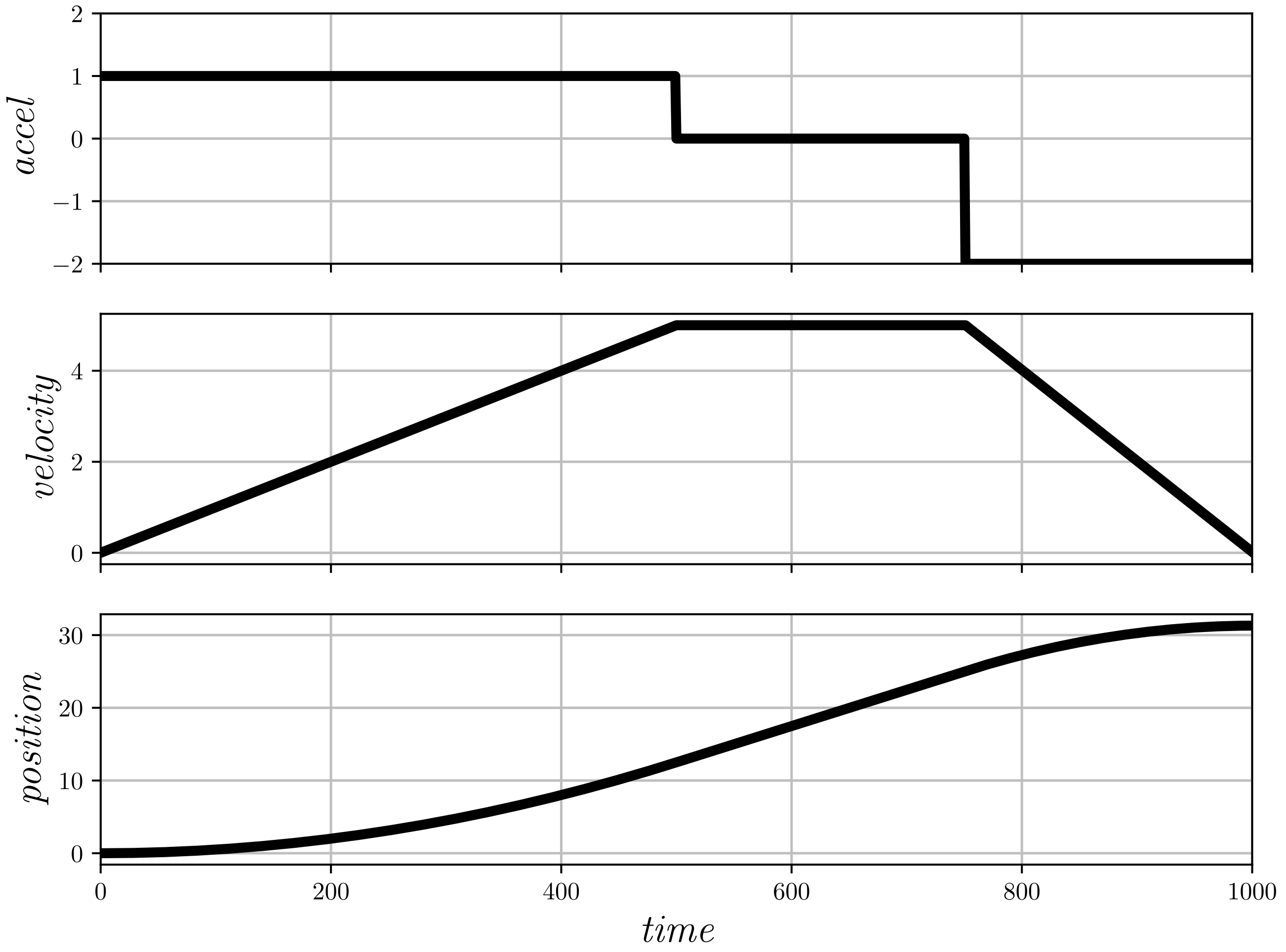
Definition 1. (First attempt) Given x_0 and u, $\xi : \mathbb{R} \to \mathbb{R}^n$ is a solution or trajectory iff

(1) $\xi(t_0) = x_0$ and

(2) $\frac{d}{dt}\xi(t) = f(\xi(t), u(t), t)), \forall t \in \mathbb{R}$.

Mathematically makes sense, but too restrictive. Assumes that ξ is not only continuous, but also differentiable. This disallows $\mathrm{u}(t)$ to be discontinuous, which is often required for optimal control.

Getting from point a to point b



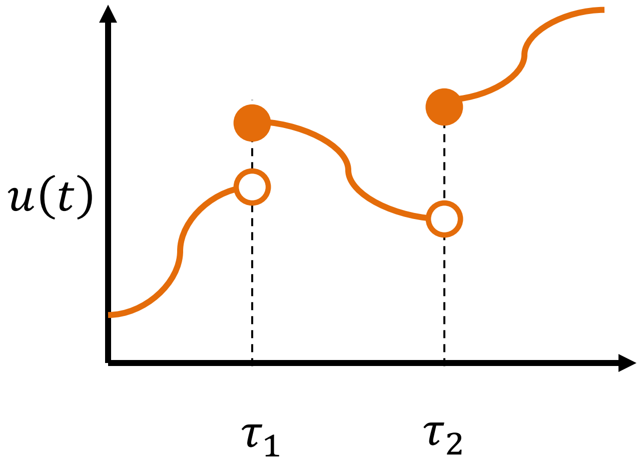
Modified notion

Definition. $u(\cdot)$ is a piece-wise continuous with set of discontinuity points $D \subseteq \mathbb{R}^m$ if

(1) $\forall \tau \in D, \lim_{t \to \tau^+} u(t) < \infty; \lim_{t \to \tau^-} u(t) < \infty$

(2) Continuous from right $\lim_{t \to \tau^+} u(t) = u(t)$

(3) $\forall\ t_0 < t_1$, $[t_0, t_1] \cap D$ is finite



$PC([t_0, t_1], \mathbb{R}^m)$ is the set of all piece-wise continuous functions over the domain $[t_0, t_1]$

Define $p(\xi(t), t) = f(x(t), u(t), t)$, for a given $u(t)$. Since $u(t)$ is PC in t so is p in the second argument.

Definition 2. Given x_0 and u, $\xi: \mathbb{R} \to \mathbb{R}^n$ is a solution or trajectory iff

(1) $\xi(t_0) = x_0$ and (2) $\frac{d}{dt}\xi(t) = p(\xi(t), t), \forall t \in \mathbb{R} \backslash \mathrm{D}$.

Is PC input $u(t)$ adequate for guaranteeing existence of solutions?

Example. $\dot{x}(t) = -sgn\big(x(t)\big); x_0 = c; t_0 = 0; c > 0$
Solution: $\xi(t) = c - t$ for $t \leq c$; check $\dot{\xi} = -1$
Problem: f discontinuous is x

Example. $\dot{x}(t) = x^2; x_0 = c; t_0 = 0; c > 0$
Solution: $\xi(t) = \frac{c}{1-tc}$ works for $t < 1/c$; check $\dot{\xi}$
Problem: As $t \rightarrow \frac{1}{c}$ then $x(t) \rightarrow \infty$; p grows too fast

Lipschitz continuity

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is Lipschitz continuous if there exist $L > 0$ such that for any pair $x, x' \in \mathbb{R}^n$, $\left|\left|f(x) - f(x')\right|\right| \leq L\left|\left|x - x'\right|\right|$

Examples: $6x + 4$; $|x|$; all differentiable functions with bounded derivatives

Non-examples: $\sqrt{x}$; x^2 (locally Lipschitz)

Existence and uniqueness of solutions

Theorem. If $p(x(t), x)$ is Lipschitz continuous in the first argument then (1) has unique solutions.

Transition system model

Linear time-varying systems

In general, for nonlinear dynamical systems we do not have closed form solutions for $\xi(t)$, but there are numerical solvers like CAPD, VNODE

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$ --- (2)
$y(t) = C(t)x(t) + D(t)u(t)$

$u(t)$ continuous everywhere except D_x

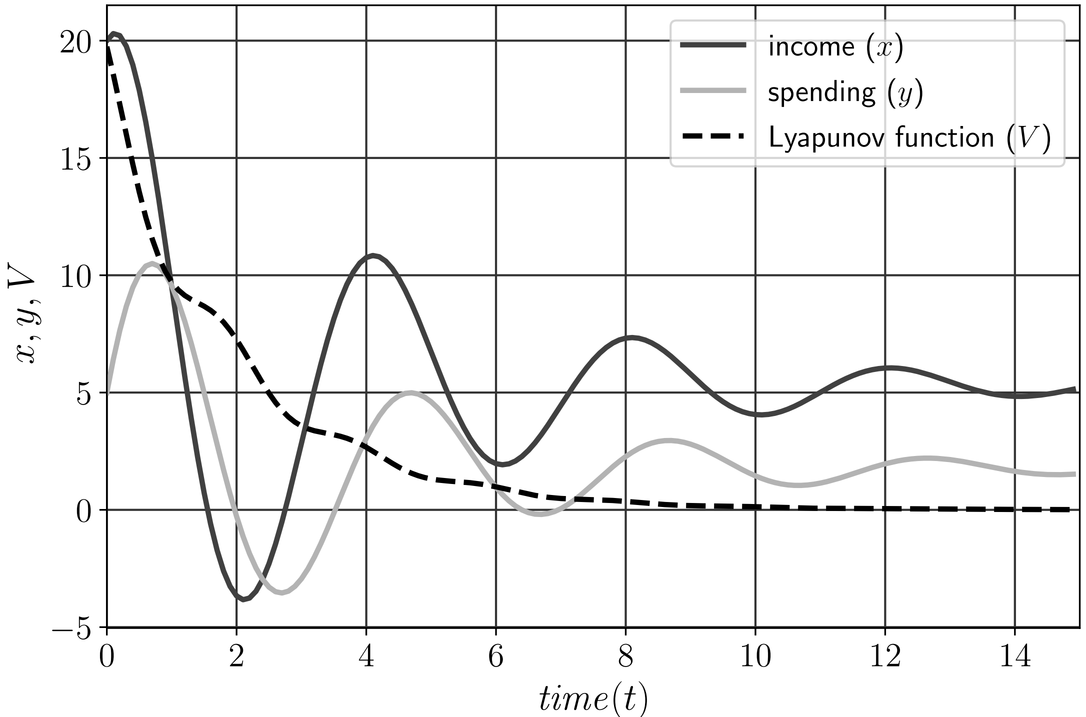
Theorem. Let $\xi(t, t_0, x_0, u)$ be the solution for (2) with points of discontinuity , D_x

1. $\forall t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(\cdot, t_0, x_0, u): \mathbb{R} \to \mathbb{R}^n$ is continuous and differentiable $\forall\ t \in \mathbb{R} \setminus D_x$
2. $\forall t, t_0 \in \mathbb{R}, u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(t, t_0, \cdot, u): \mathbb{R}^n \to \mathbb{R}^n$ is continuous
3. $\forall t, t_0 \in \mathbb{R}, x_{01}, x_{02} \in \mathbb{R}^n, u_1, u_2 \in PC(\mathbb{R}, \mathbb{R}^m), a_1, a_2 \in \mathbb{R}, \xi(t, t_0, a_1 x_{01} + a_2 x_{02}, a_1 u_1 + a_2 u_2) = a_1 \xi(t, t_0, x_{01}, u_1) + a_2 \xi(t, t_0, x_{02}, u_2)$
4. $\forall t, t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(t, t_0, x_0, u) = \xi(t, t_0, x_0, \mathbf{0}) + \xi(t, t_0, 0, u)$

Example 1: Simple model of an economy

- x: national income
- y: rate of consumer spending
- g: rate government expenditure

- $\dot{x} = x - \alpha y$
- $\dot{y} = \beta(x - y - g)$



Linear system and solutions

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$

For a given initial state $x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R} and\ u(.) \in PC(\mathbb{R}, \mathbb{R}^n)$ the *solution* is a function $\xi(., t_0, x_0, u): \mathbb{R} \rightarrow \mathbb{R}^n$

We studied several properties of ξ in the last lecture: continuity with respect to first and third argument, linearity, decomposition

Linear system and solutions

- Since $\xi(.,t_0,x_0,u)\colon \mathbb{R} \to \mathbb{R}^n$ is a linear function of the initial state and input,
- $\xi(t,t_0,x_0,u)$ = $\xi(t,t_0,0,u)$ + $\xi(.,t_0,x_0,0)$
- Let us focus on the linear function $\xi(.,t_0,x_0,0)$
- Define $\Phi(.,t_0)x_0 = \xi(.,t_0,x_0,u)$
- This $\Phi(.,t_0)\colon\mathbb{R} \to \mathbb{R}^{n\times n}$ is called the <u>state transition matrix</u>

Properties of Φ

- $\Phi(.,t_0): \mathbb{R} \to \mathbb{R}^{n\times \mathrm{n}}$ is the unique solution of (2) and is defined by a (Peano-Baker) infinite sequence of integrals
- $\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0)$ with $\Phi(t,t) = I$
  - Continuous everywhere
  - Differentiable everywhere except D_x ($A(t)$ isn't)
- $\forall t_0, t_1, t\ \Phi(t,t_0) = \Phi(t,t_1)\Phi(t_1,t_0)$
- $\Phi(t,t_0)$ is invertible $[\Phi(t,t_0)]^{-1} = \Phi(t_0,t)$

Solution of linear systems in Φ

Theorem.

$$\xi(t, t_0, x_0, u) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

Linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Matrix exponential:

$$e^{At} = 1 + At + \frac{1}{2!}(At)^2 + \ \dots = \sum_{0}^{\infty} \frac{1}{k!}(At)^k$$

Theorem. $\Phi(t, t_0) = e^{A(t-t_0)}$, that is

$$\xi(t, t_0, x_0, u) = x_0 \mathrm{e}^{A(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} Bu(\tau) d\tau$$

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Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*

Cleve Moler†
Charles Van Loan‡

Abstract. In principle, the exponential of a matrix could be computed in many ways. Methods involving approximation theory, differential equations, the matrix eigenvalues, and the matrix characteristic polynomial have been proposed. In practice, consideration of computational stability and efficiency indicates that some of the methods are preferable to others, but that none are completely satisfactory.
Most of this paper was originally published in 1978. An update, with a separate bibliography, describes a few recent developments.

Key words. matrix, exponential, roundoff error, truncation error, condition

AMS subject classifications. 15A15, 65F15, 65F30, 65L99

PII. S0036144502418010

Discrete time models / discrete transition systems

- $x(t+1) = f\big(x(t), u(t)\big)$
- $x(t+1) = f(x(t))$ autonomous
- Execution: $x_0, f(x_0), f^2(x_0), \ldots$
- $\boldsymbol{A} = \langle Q, Q_0, T \rangle$
  - $Q = \mathbb{R}^n, Q_0 = \{x_0\}$
  - $T: \mathbb{R}^n \to \mathbb{R}^n; \mathrm{T}(x) = f(x)$
- Deterministic

Discretized or sampled-time model

- $\dot{x}(t) = f\big(x(t), u(t)\big)$
- Assume: $u \in PC(\mathbb{R}, U)\ where\ U \subseteq \mathbb{R}^m$ is a finite set
- $\xi(t, t_0, x_0, u)$
- Fix a sampling period $\delta > 0$
- $\boldsymbol{A_\delta} = \langle Q, Q_0, U, T\rangle$
  - $Q = \mathbb{R}^n, Q_0 = \{x_0\}, Act = U,$
  - $T \subseteq \mathbb{R}^n \times U \times \mathbb{R}^n; (x, u, x') \in \mathrm{T}\ iff\ x' = \xi(\delta, 0, x, u)$

Properties for dynamical systems

What type of properties are we interested in?

- Invariance (as in the case of automata)
- State remains bounded
- Converges to target
- Bounded input gives bounded output (BIBO)

Requirements: Stability

- We will focus on time invariant autonomous systems (closed systems, systems without inputs)
- $\dot{x}(t) = f\big(x(t)\big), x_0 \in \mathbb{R}^n, t_0 = 0$ *–(1)*
- $\xi(t)$ is the solution
- $|\xi(t)|$ norm
- $x^* \in \mathbb{R}^n$ is an **equilibrium point** if $f(x^*) = 0$.
- For analysis we will assume **0** to be an equilibrium point of (1) with out loss of generality

Example: Pendulum

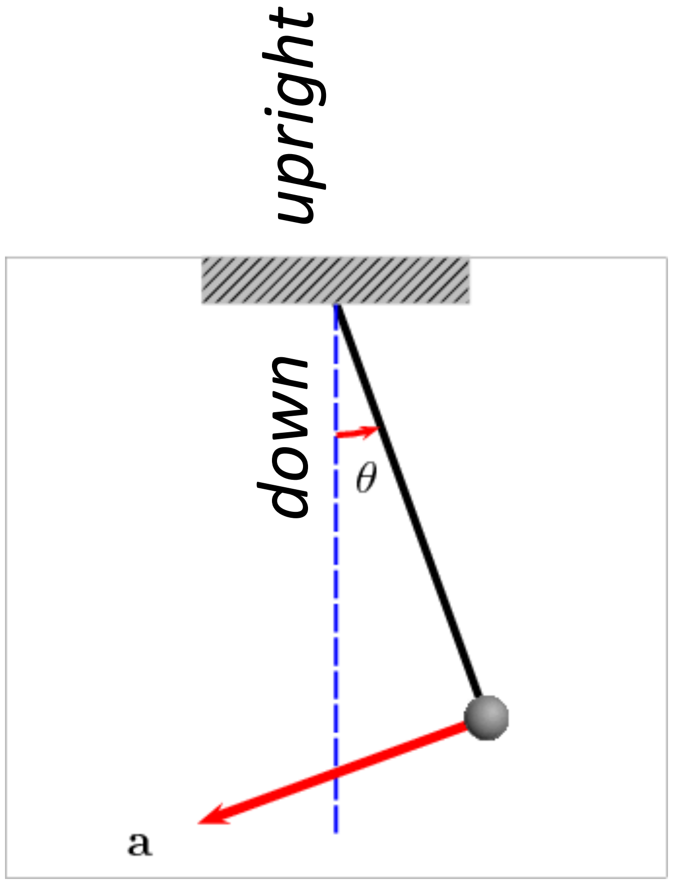
Pendulum equation

$x_1 = \theta \quad x_2 = \dot{\theta}$

$x_2 = \dot{x}_1$

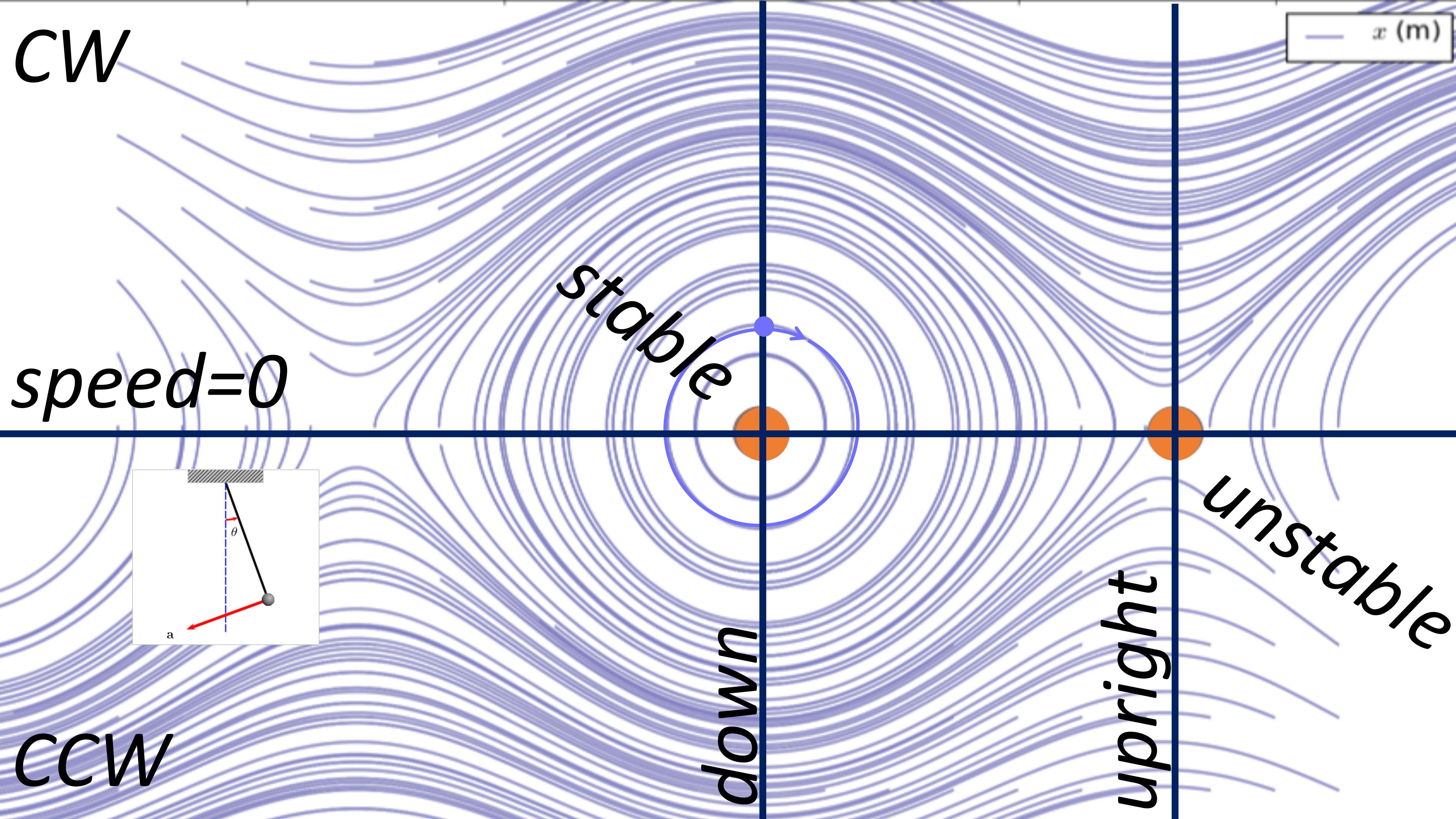
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \\ x_2 \end{bmatrix}$$



k: friction coefficient

Two equilibrium points: $(0,0), (\pi, 0)$


CW
x (m)
speed=0
stable
unstable
θ
a
down
upright
CCW

Aleksandr M. Lyapunov

Aleksandr Mikhailovich Lyapunov (June 6 1857–November 3, 1918) was a Russian mathematician and physicist.

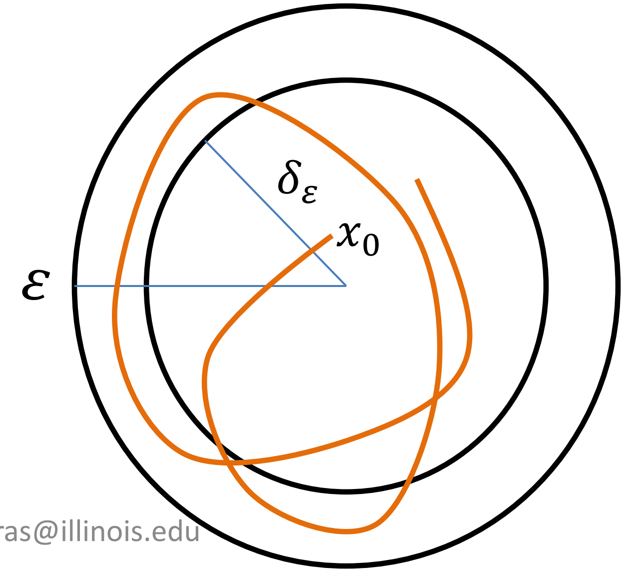
His methods make it possible to define the stability of ordinary differential equations. In the theory of probability, he generalized the works of Chebyshev and Markov, and proved the Central Limit Theorem under more general conditions than his predecessors.

Lyapunov stability

Lyapunov stability: The system (1) is said to be ***Lyapunov stable*** (at the origin) if

$$\forall \varepsilon > 0 \; \exists \; \delta_\varepsilon > 0 \text{ such that } |x_0| \leq \delta_\varepsilon \Rightarrow \forall \, t \geq 0, |\xi(x_0, t)| \leq \varepsilon.$$

How is this related to invariants and reachable states ?

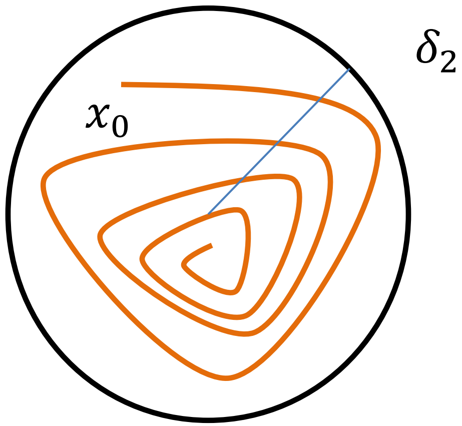


Asymptotically stability

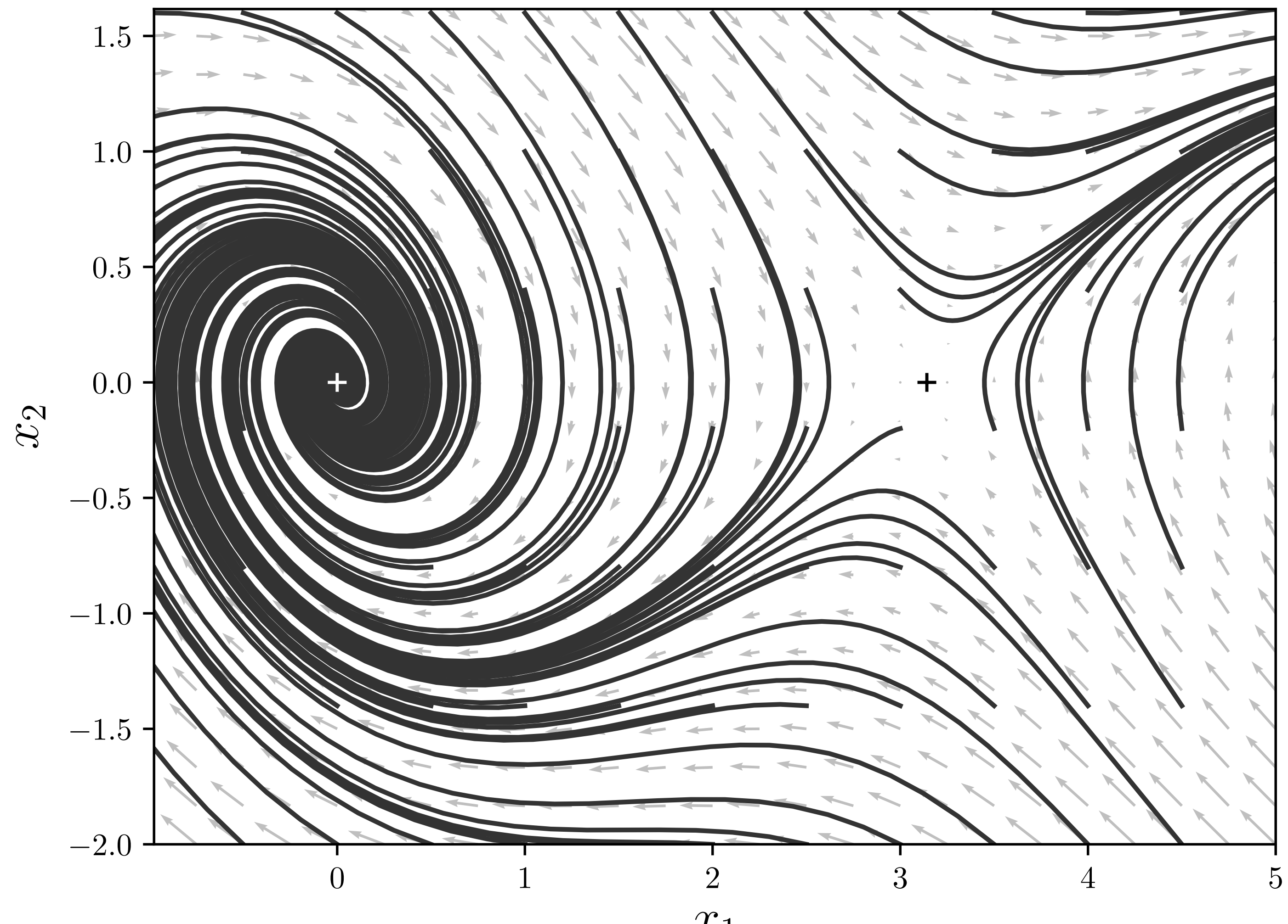
The system (1) is said to be ***Asymptotically stable*** *(at the origin)* if it is Lyapunov stable and

$\exists \delta_2 > 0$ such that $\forall |x_0| \leq \delta_2$ as t $\rightarrow \infty, |\xi(x_0, t)| \rightarrow \mathbf{0}$.

If the property holds for any δ_2 then **Globally Asymptotically Stable**



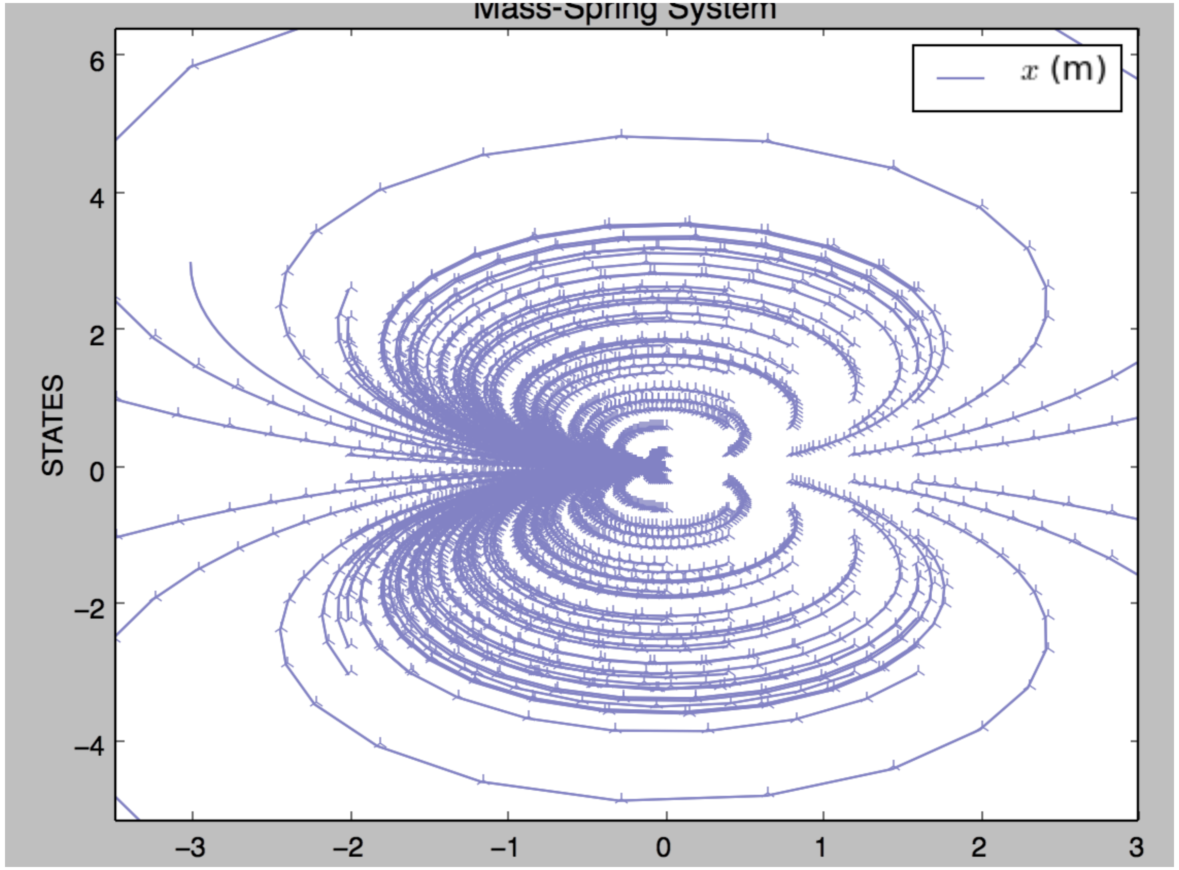
Phase portrait of pendulum with friction



Butterfly*

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 2x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix}$$

All solutions converge to 0 but the equilibrium point (0,0) is not Lyapunov stable



*Not discussed in class

Van der pol oscillator

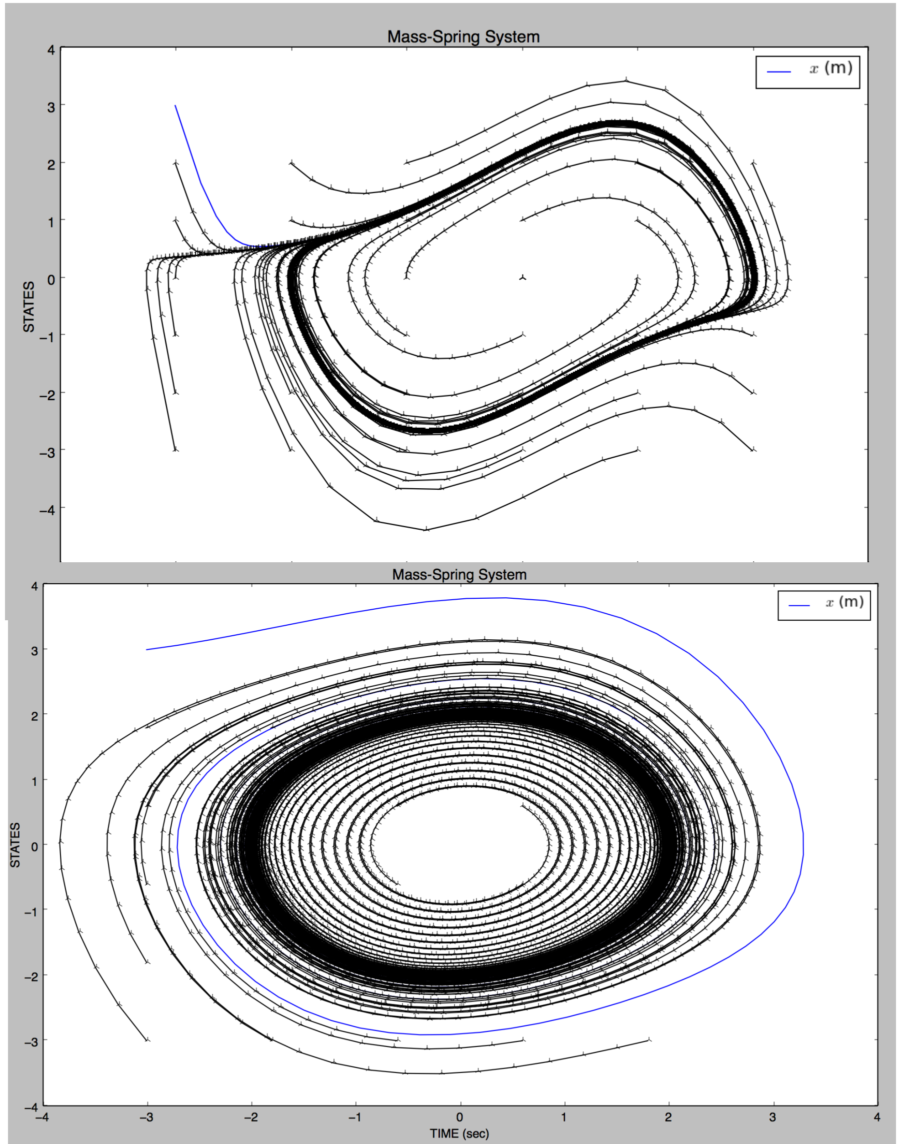
Van der pol oscillator

$$\frac{dx^2}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

$x_1 = x; x_2 = \dot{x}_1;$
coupling coefficient $\mu = 2 \;\; 0.1$

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} \mu(1 - x_1^2)x_2 - \ x_1 \\ x_2 \end{bmatrix}$$

stable ?



Stability of solutions* (instead of points)

- For any $\xi \in \mathrm{PC}(\mathbb{R}^{\geq 0}, \mathbb{R}^n)$ define the s-norm $\left||\xi|\right|_s = \sup_{t\in\mathbb{R}} ||\xi(t)||$
- A dynamical system can be seen as an operator that maps initial states to signals $T: \mathbb{R}^n \to PC(\mathbb{R}^{\geq 0}, \mathbb{R}^n)$
- Lyapunov stability required that this operator is continuous
- The solution ξ^* is ***Lyapunov stable*** if T is continuous as $\xi^*(0)$. $\mathrm{i.e.}$, for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every $x_0 \in \mathbb{R}^n$ if $|\xi^*(0) - x_0| \leq \delta_\varepsilon$ then $\left|\left|T\left(\xi^*(t)\right) - T(x_0)\right|\right|_s \leq \varepsilon$.

*Not discussed in class

Verifying Stability for Linear Systems

Consider the linear system $\dot{x} = Ax$

Theorem.

1. It is asymptotically stable iff all the eigenvalues of *A* have **strictly** negative real parts (*Hurwitz*).

2. It is Lyapunov stable iff all the eigenvalues of *A* have real parts that are either zero or negative and the *Jordan blocks* corresponding to the eigenvalues with zero real parts are of size 1.

Jordan decomposition

For every *n x n* matrix *A*, there exists a nonsingular *n x n* matrix *P* such that

$$PAP^{-1} = J = \begin{bmatrix} J_1 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ 0 & 0 & J_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & J_\ell \end{bmatrix},$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ 0 & 0 & \lambda_i & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i \end{bmatrix}.$$

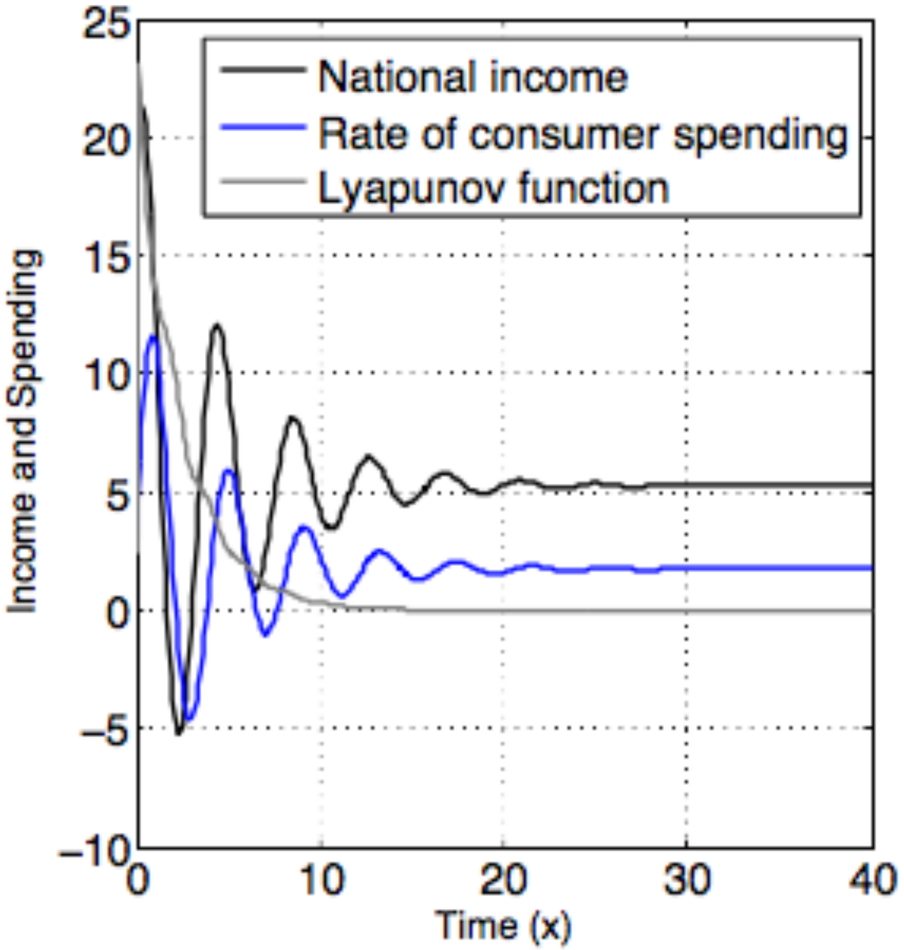
where each J_i is a upper triangular matrix called a Jordan block

Example 1: Simple model of an economy

- x: national income y: rate of consumer spending; g: rate government expenditure
- $\dot{x} = x \ - \alpha y$
- $\dot{y} = \beta(x - y - g)$
- $g = g_0 + kx$ $\qquad\qquad\qquad$ α, β, k are positive constants
- What is the equilibrium?
- $x^* = \frac{g_0 \alpha}{\alpha - 1 - k\alpha} \, y^* = \frac{g_0 \alpha}{\alpha - 1 - k\alpha}$
- Dynamics:
- $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \beta(1-k) & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Example: Simple linear model of an economy

- $\alpha = 3, \beta = 1, k = 0$
- $\lambda_1, \lambda_1^* = (-.25 \pm i\ 1.714)$
- Negative real parts, therefore, asymptotically stable and the national income and consumer spending rate converge to $x = 1.764\ y = 5.294$



Stability of nonlinear systems

- For any ***positive definite*** function of state $V: \mathbb{R}^n \to \mathbb{R}$
  - $V(x) \geq 0$ and $V(x) = 0$ iff $x = 0$
- ***Sublevel sets*** of $L_p = \{x \in \mathbb{R}^n \mid V(x) \leq p\}$
- $V(\xi(t))$

V differentiable with continuous first derivative

- $\dot{V} = d\frac{V(\xi(t))}{dt} = ?$
- $\frac{\partial V}{\partial x} . \frac{d}{dt}(\xi(t)) = \frac{\partial V}{\partial x} . f(x)$ is also continuous
- V is radially unbounded if $||x|| \to \infty \Rightarrow V(x) \to \infty$

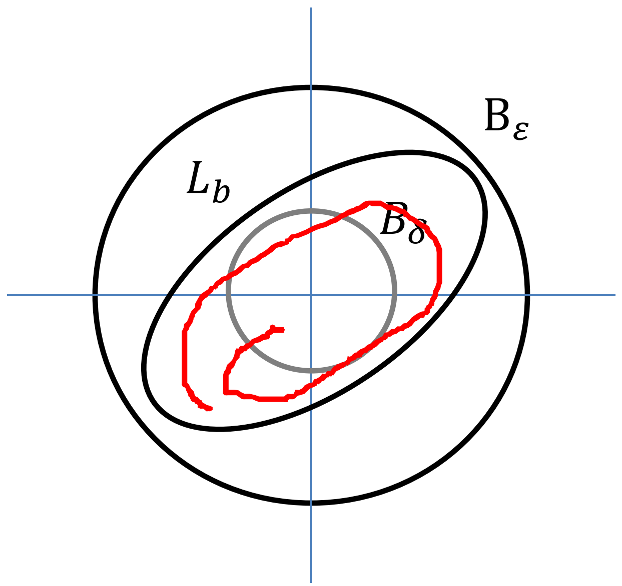
Verifying Stability

Theorem. (Lyapunov) Consider the system (1) with state space $\xi(t) \in \mathbb{R}^n$ and suppose there exists a positive definite, continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$. The system is:

1. Lyapunov stable if $\dot{V}\big(\xi(t)\big) = \frac{\partial V}{\partial x} f(x) \leq 0$, for all $x \neq 0$
2. Asymptotically stable if $\dot{V}\big(\xi(t)\big) < 0$, for all $x \neq 0$
3. It is globally AS if *V* is also radially unbounded.

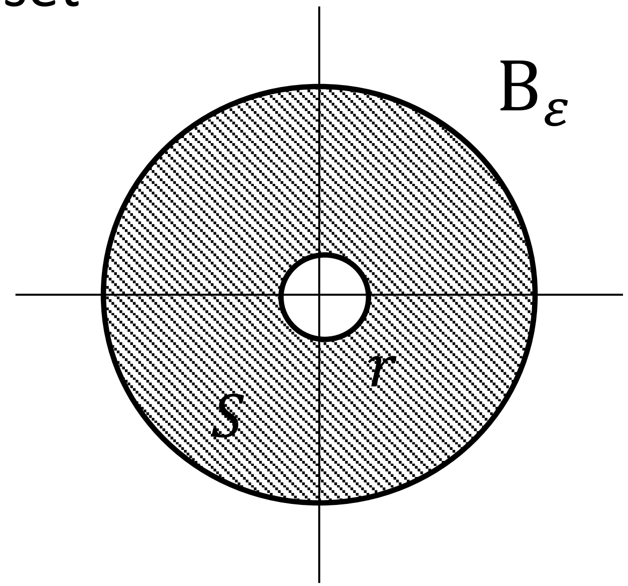
Proof sketch: Lyapunov stable if $\dot{V} \leq 0$

- Assume $\dot{V} \leq 0$
- Consider a ball B_ε around the origin of radius $\varepsilon > 0$.
- Pick a positive number $b < \min_{|x|=\varepsilon} V(x)$.
- Let δ be a radius of ball around origin which is inside $\mathrm{B}_\delta = \{x \,|V(x) \leq b\}$
- Since along all trajectories V is non-increasing, starting from B_δ each solution satisfies $V\big(\xi(t)\big) \leq b$ and therefore remains in B_ε



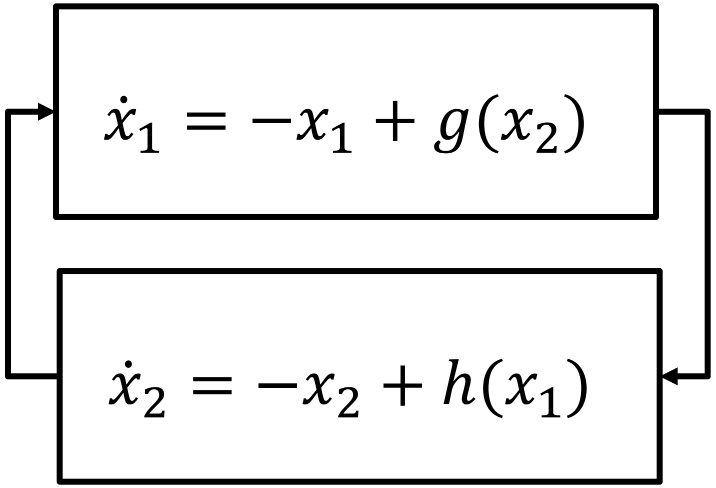
Proof sketch: Asymptotically stable if $\dot{V}\big(\xi(t)\big) < 0$

- Assume $\dot{V} < 0$
- Take arbitrary initial state $|\xi(0)| \leq \delta$, where this δ comes from some ε for Lyapunov stability
- Since $V\big(\xi(.)\big) > 0$ and decreasing along ξ it has a limit $c \geq 0$ at $t \rightarrow \infty$
- It suffices to show that this limit is actually 0
- Suppose not, c > 0 then the solution $\xi(0)$ evolves in the compact set $S = \{x \mid r \leq |x| \leq \varepsilon\}$ for some sufficiently small r
- Let $d = \max_{x \in S} \dot{V}(x)$ [slowest rate]
- This number is well-defined and negative
- $\dot{V}\big(\xi(t)\big) \leq d$ for all t
- $V(t) \leq V(0) + dt$
- But then eventually $V(t) < c$



Example 2: Reasoning about stability without solving ODEs

$\dot{x}_1 = -x_1 + g(x_2); \dot{x}_2 = -x_2 + h(x_1)$



Given that $|g(x_2)| \leq \frac{|x_2|}{2}, |h(x_1)| \leq \frac{|x_1|}{2}$

- Use $V = \frac{1}{2}(x_1^2 + x_2^2) \geq 0$
- $\dot{V} = x_1\,\dot{x}_1 + x_2\,\dot{x}_2$

$$= -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1)$$

$$\leq -x_1^2 - x_2^2 + \frac{1}{2}(|x_1 x_2| + |x_2 x_1|)$$

$$\leq -\frac{1}{2}(x_1^2 + x_2^2) = -V$$

$$(|x_1| - |x_2|)^2 \geq 0$$

$$x_1^2 + x_2^2 \geq 2|x_1 x_2|$$

$$|x_1 x_2| \leq \frac{1}{2}(x_1^2 + x_2^2)$$

We conclude global asymptotic stability (in fact global exponential stability) without knowing solutions

Proposition. If V is a Lyapunov function then every sublevel set of V is an invariant

Proof. $V\big(\xi(t)\big) =$

$$= V\big(\xi(0)\big) + \int_0^t \dot{V}\big(\xi(\tau)\big)d\tau$$
$$\leq V(\xi(0))$$

An aside: Checking inductive invariants

- $\boldsymbol{A} = \langle X, Q_0, T \rangle$
  - X: set of variables
  - $Q_0 \subseteq val(X)$
  - $T \subseteq val(X) \times val(X)$ written as a program $x' \subseteq T(x)$
- How do we check that $I \subseteq val(X)$ is an inductive invariant?
  - $Q_0 \Rightarrow I(X)$
  - $I(X) \Rightarrow I(T(X))$
- Implies that $Reach_{\boldsymbol{A}}(Q_0) \subseteq I$ without computing the executions or reachable states of $\boldsymbol{A}$
- The key is to find such I

Finding Lyapunov Functions

- The key to using Lyapunov theory is to *find* a Lyapunov function and verify that it has the properties
- In general, for nonlinear systems this is hard
- There are several approaches
  - Quadratic Lyapunov functions for linear systems
  - Decide the form/template of the function (e.g., quadratic, polynomial), parameterized by some parameters and find values of the parameters so that the conditions hold (Chapter 3 last section)

Linear autonomous systems

- $\dot{x} = Ax, A \in \mathbb{R}^{n \times n}$
- The Lyapunov equation: $A^T P + PA + Q = 0$

  where $P, Q \in \mathbb{R}^{n \times n}$ are symmetric

- Interpretation: $V(x) = x^T P x$ then

$$\dot{V}(x) = (Ax)^T P x + x^T P(Ax)$$

[using chain rule $\frac{\partial u^T P v}{\partial t} = \frac{\partial u}{\partial t} P v + \frac{\partial v}{\partial t} P^T u$]

$$= x^T (A^T P + PA) x = -x^T Q x$$

- If $x^T P x$ is the generalized energy then $-x^T Q x$ is the associated dissipation

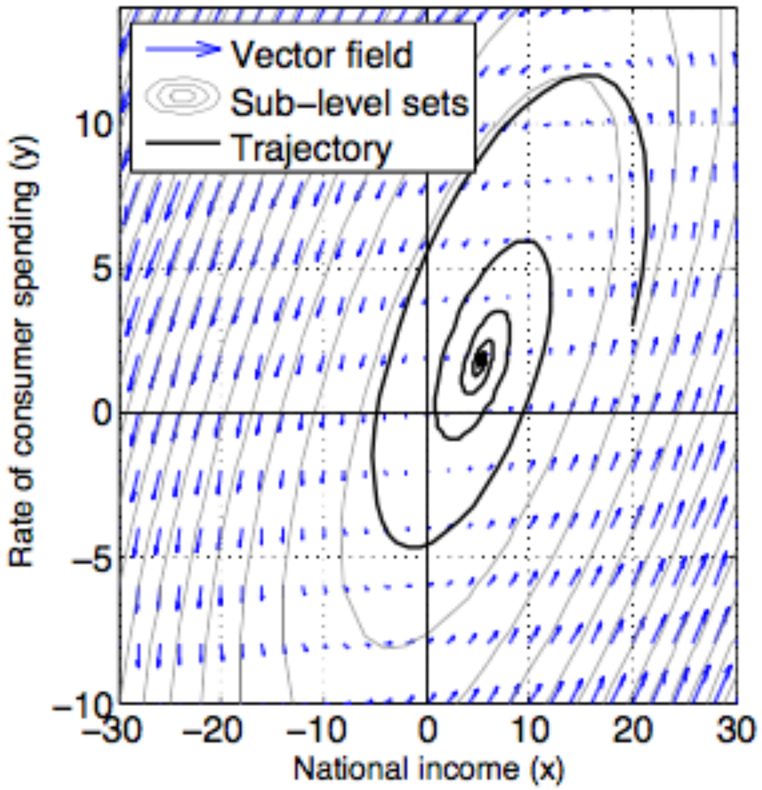
Quadratic Lyapunov Functions

- If $P > 0$ (positive definite)
- $V(x) = x^T P x = 0 \Leftrightarrow x = 0$
- The sub-level sets are ellipsoids
- If $Q > 0$ then the system is globally asymptotically stable

Same example

Lyapunov equations are solved as a set of $\frac{n(n+1)}{2}$ equations in $n(n+1)/2$ variables. Cost O(n^6)

Choose $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ solving Lyapunov equations we get $P = \begin{bmatrix} 2.59 & -2.29 \\ -2.29 & 4.92 \end{bmatrix}$ and we get the quadratic Lyapunov function $(x - x^*)P(x - x^*)^T$ an a sequence of invariants



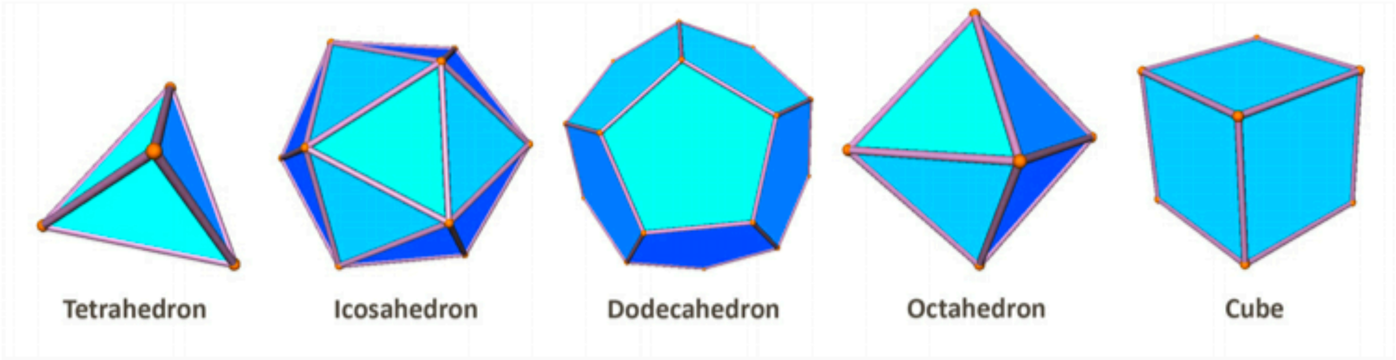
Converse Lyapunov

Converse Lyapunov theorems show that conditions of the previous theorem are also necessary. For example, if the system is asymptotically stable then there exists a positive definite, continuously differentiable function V, that satisfies the inequalities.

For example if the LTI system $\dot{x} = Ax$ is globally asymptotically stable then there is a quadratic Lyapunov function that proves it.

Small puzzle

- Platonic solids. Solid bodies whose faces are regular polygons, all identical, that meet in identical fashion at every vertex. How many such are there? Exactly five!



Platonic solids.
Wilczek, Frank. A Beautiful Question: Finding Nature's Deep Design (p. 39).