Verifying Dynamical Systems: Stability Part 2

Sayan Mitra

Verifying cyberphysical systems

mitras@illinois.edu

Plan

- Dynamical system models
  - Lyapunov method
  - Proofs
  - Examples
  - Stability of linear systems

Requirements: Stability

- We will focus on time invariant autonomous systems (closed systems, systems without inputs)
- $\dot{x}(t) = f\big(x(t)\big), x_0 \in \mathbb{R}^n, t_0 = 0$ *–(1)*
- $x(t)$ is the solution
- $|x(t)|$ norm
- $x^* \in \mathbb{R}^n$ is an **equilibrium point** if $f(x^*) = 0$.
- For analysis we will assume $x^* = \mathbf{0}$ to be an equilibrium point of (1) with out loss of generality

Example: Pendulum

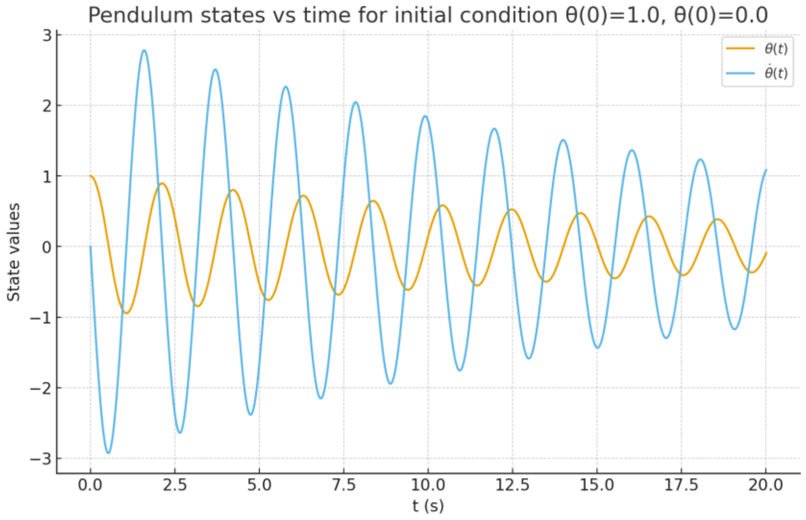
Pendulum equation

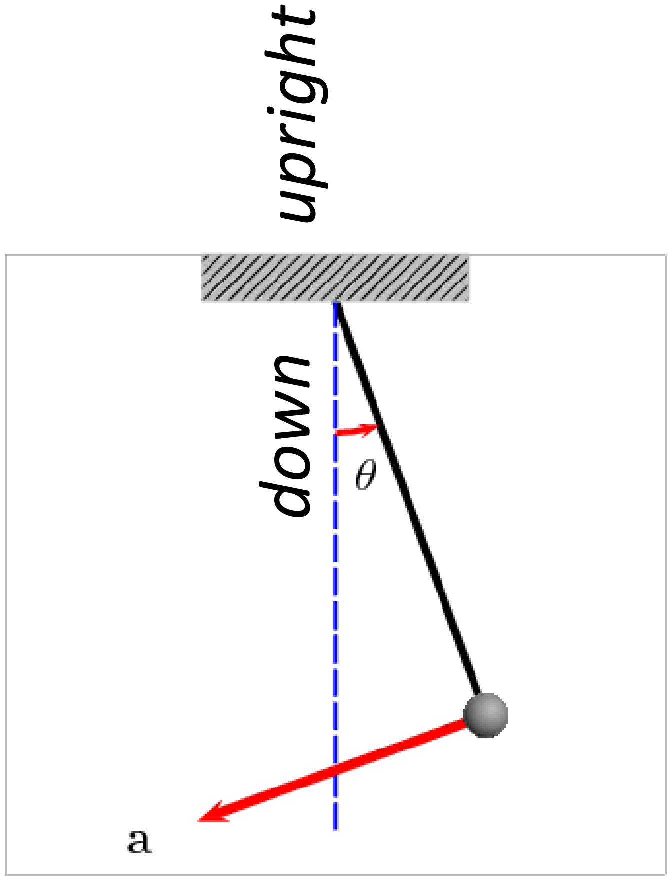
$x_1 = \theta \;\; x_2 = \dot{\theta}$

$x_2 = \dot{x}_1$

$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

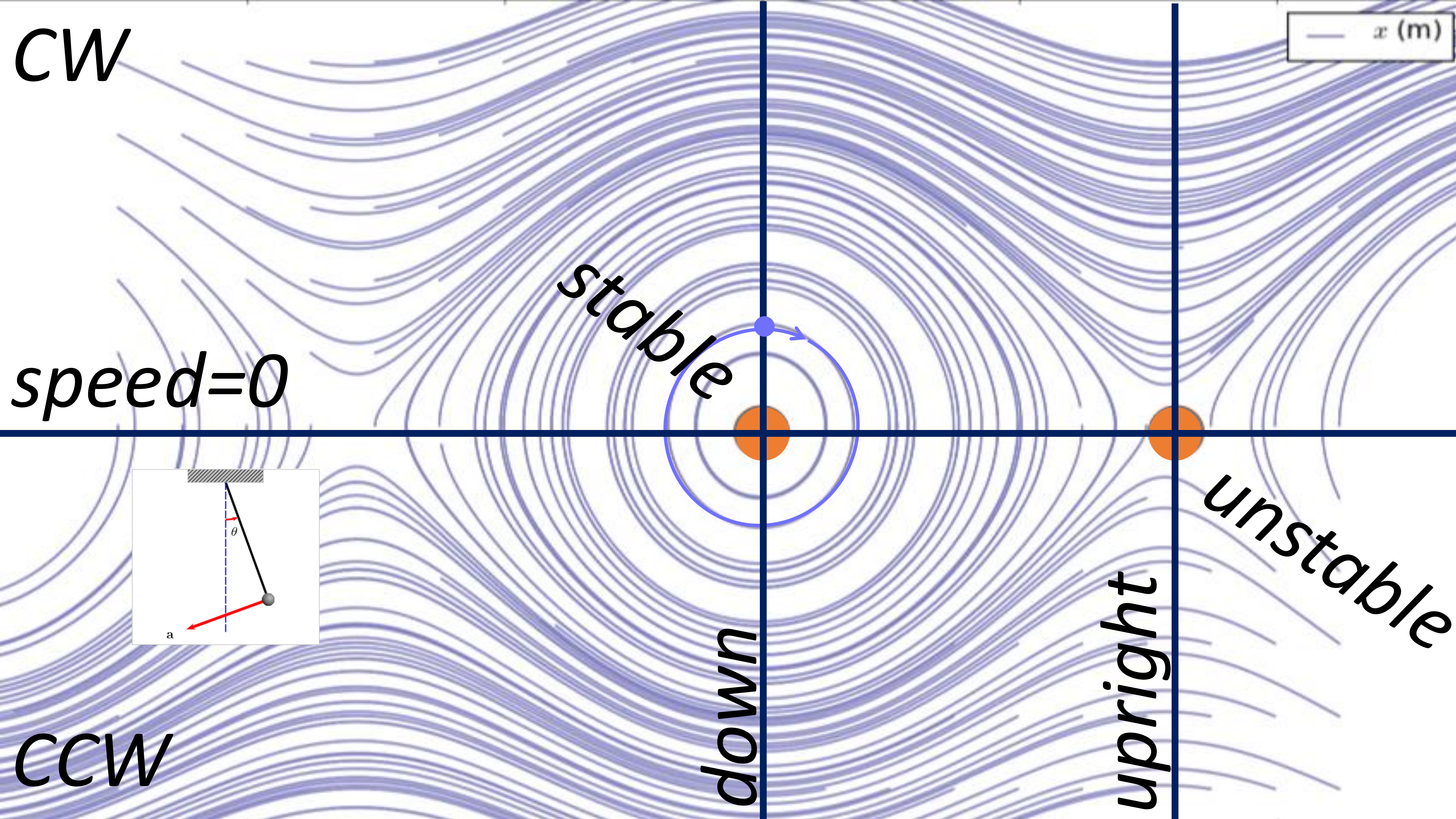
$$\begin{bmatrix}\dot{x}_2 \\ \dot{x}_1\end{bmatrix} = \begin{bmatrix}-\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \\ x_2\end{bmatrix}$$





k: friction coefficient

Two equilibrium points: $(0,0), (\pi, 0)$



CW
speed=0
CCW
stable
unstable
down
upright
x (m)
θ
a

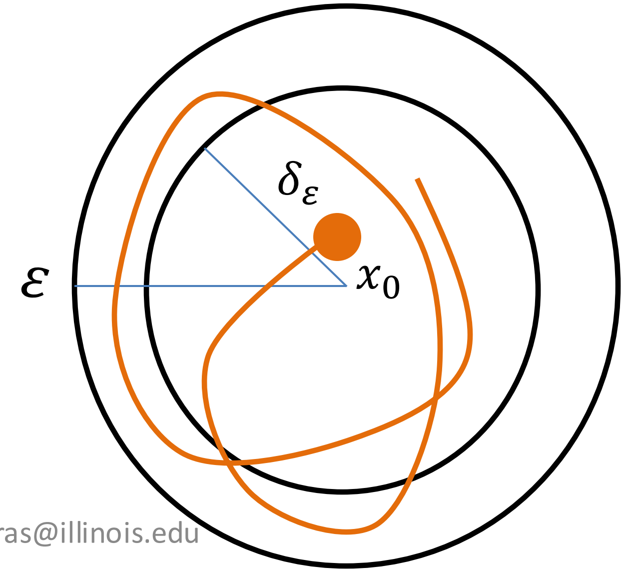
Lyapunov stability

Lyapunov stability: The system (1) is said to be ***Lyapunov stable*** (at the origin) if

$\forall \varepsilon > 0 \; \exists \; \delta_\varepsilon > 0$ such that

$$|x_0| \leq \delta_\varepsilon \Rightarrow \forall \; t \geq 0, |\xi(x_0, t)| \leq \varepsilon.$$

How is this related to invariants and reachable states ?

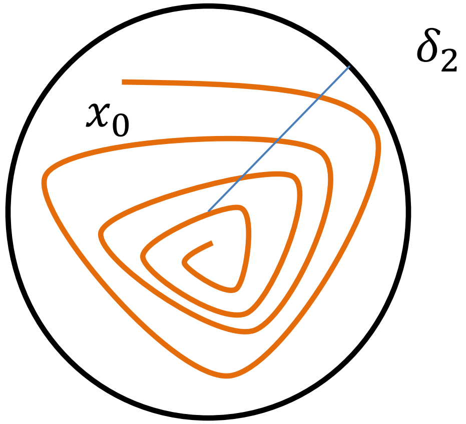


Asymptotically stability

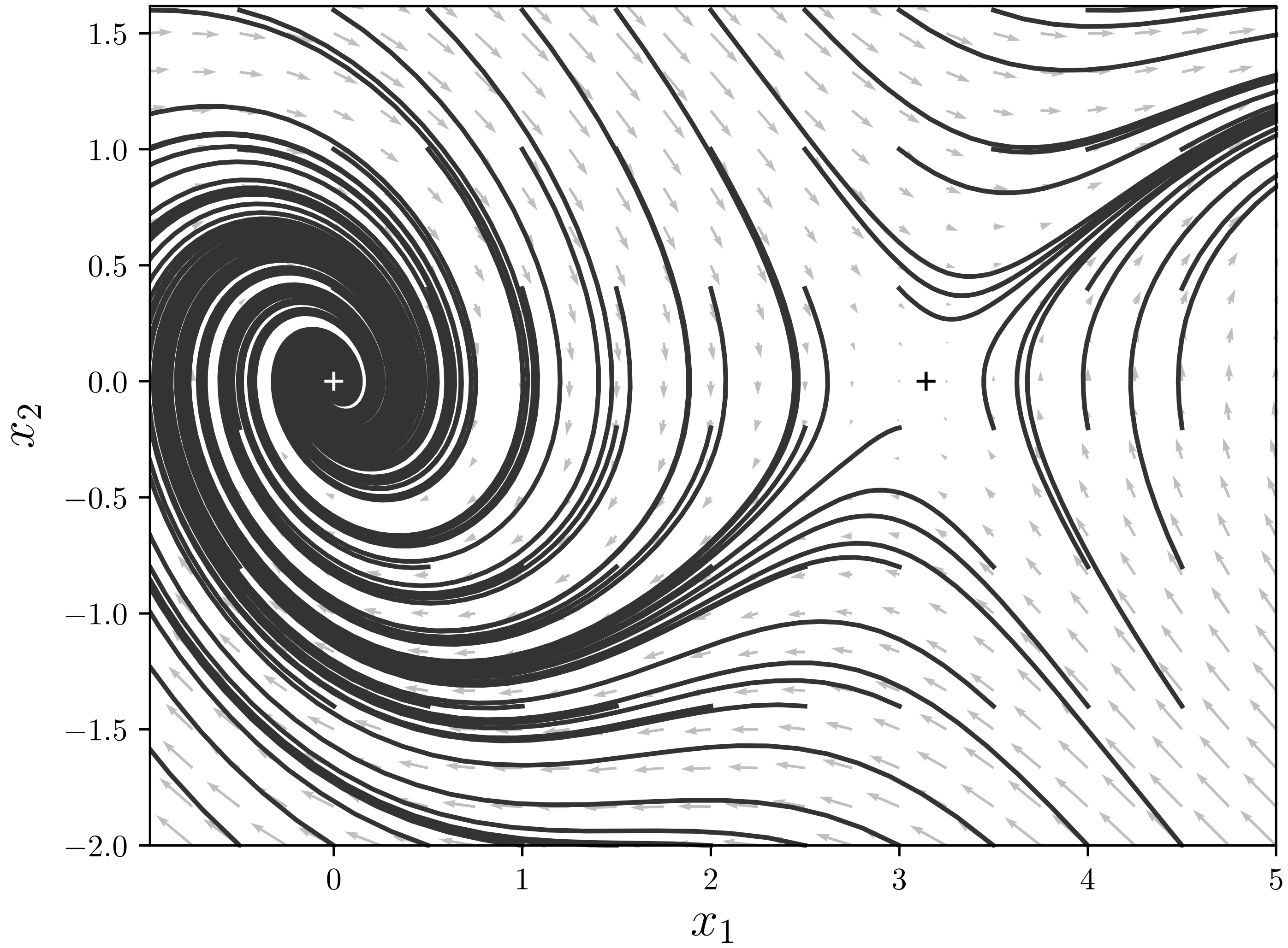
The system (1) is said to be ***Asymptotically stable*** *(at the origin)* if it is Lyapunov stable and

$\exists \delta_2 > 0$ such that $\forall |x_0| \leq \delta_2$ as $t \rightarrow \infty, |\xi(x_0, t)| \rightarrow \mathbf{0}$.

If the property holds for any δ_2 then **Globally Asymptotically Stable**



Phase portrait of pendulum with friction

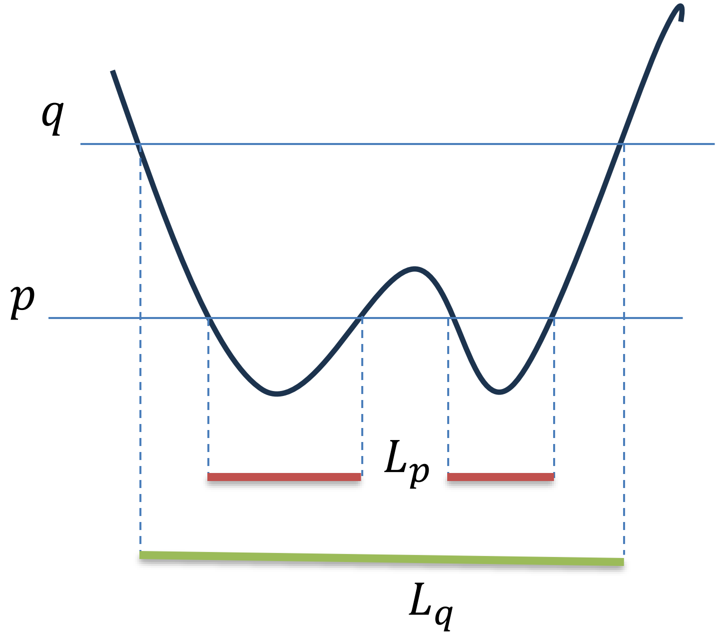


Stability of nonlinear systems

- For any ***positive definite*** function of state $V: \mathbb{R}^n \rightarrow \mathbb{R}$
  - $V(x) \geq 0$ and $V(x) = 0$ iff $x = 0$
- ***Sublevel sets*** of $L_p = \{x \in \mathbb{R}^n \mid V(x) \leq p\}$
- $V(x(t))$

V differentiable with continuous first derivative

- $\dot{V} = d\frac{V(x(t))}{dt} = ?$
- $\frac{\partial V}{\partial x} . \frac{d}{dt}(x(t)) = \frac{\partial V}{\partial x} . f(x)$ is also continuous
- V is **radially unbounded** if $||x|| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$



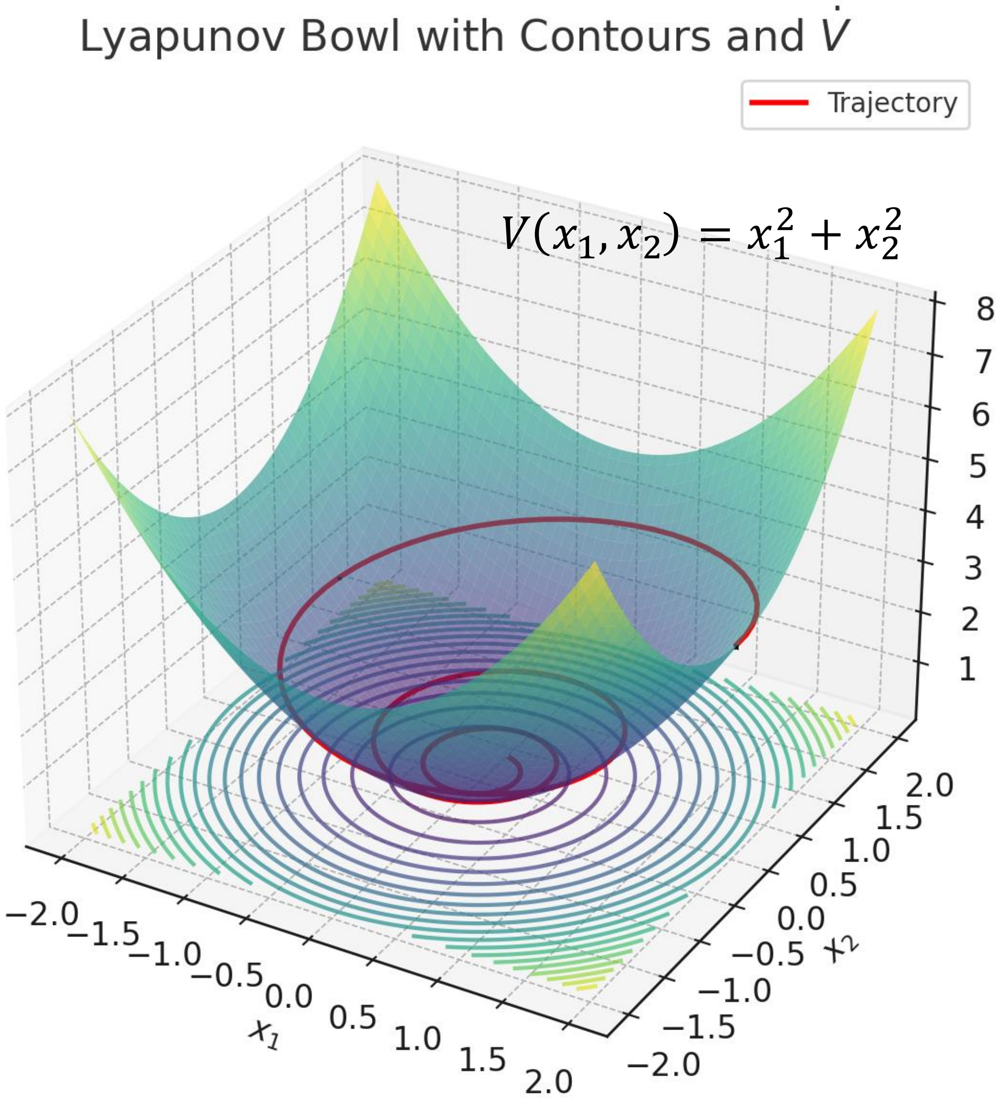
Verifying Stability

Theorem. (Lyapunov) Consider the system (1) with stat space $x(t) \in \mathbb{R}^n$ and suppose there exists a **positive definite, continuously differentiable** function $V: \mathbb{R}^n \to \mathbb{R}$. The system is:

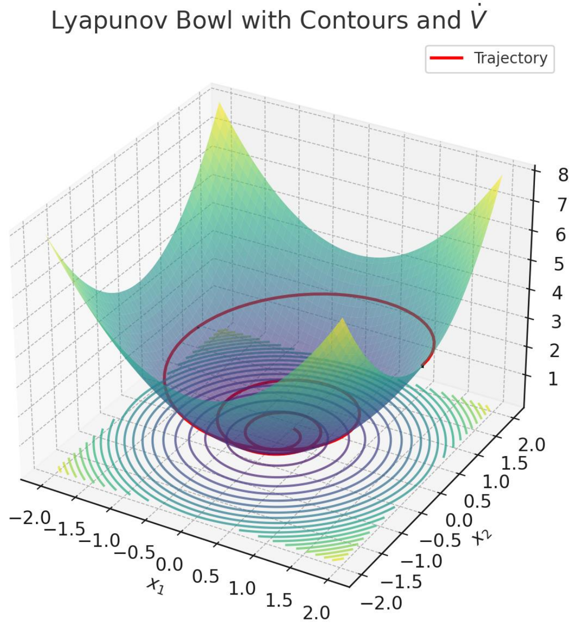
1. Lyapunov stable if $\dot{V}\big(x(t)\big) = \frac{\partial V}{\partial x} f(x) \leq 0, \ \forall x \neq 0$

2. Asymptotically stable if $\dot{V}\big(x(t)\big) < 0$, for all $x \neq 0$

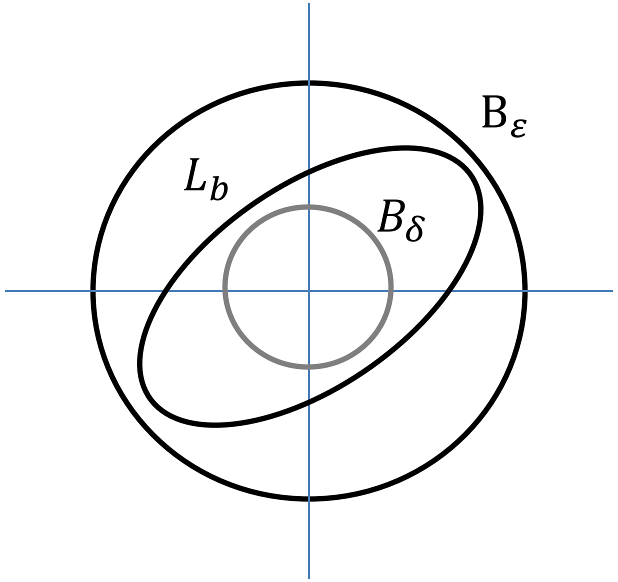
3. It is globally AS if V is also radially unbounded.



Proof : Lyapunov stable if $\dot{V} \leq 0$

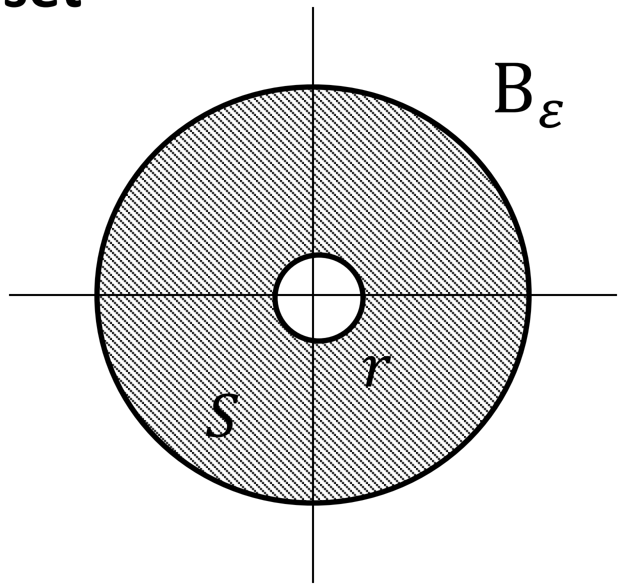


- Assume $\dot{V} \leq 0$
- Fix a ball B_ε around the origin of radius $\varepsilon > 0$.
- Pick a positive number $b < \min_{|x|=\varepsilon} V(x)$.
- Sublevel set $L_b = \{x \mid V(x) \leq b\} \subseteq B_\varepsilon$
- Claim. L_b is an invariant
  - For any $x_0 \in L_b$ $V(x(t)) \leq V(x_0) \leq b$
  - Therefore $x(t) \in L_b$
- Choose $B_\delta \subseteq L_b$
- For any $x_0 \in B_\delta \subseteq L_b$ we have $x(t) \in L_b \subseteq B_\varepsilon$



Proof sketch: Asymptotically stable if $\dot{V}\big(x(t)\big) < 0$

- Assume $\dot{V} < 0$
- Take arbitrary initial state $|x(0)| \leq \delta$, where this δ comes from some ε for Lyapunov stability
- Since $V\big(x(.)\big) > 0$ and decreasing along x it has a limit $c \geq 0$ at $t \to \infty$
- It suffices to show that this limit is actually 0
- Suppose not, c > 0 then the solution $x(0)$ evolves in the **compact set** $S = \{x \mid r \leq |x| \leq \varepsilon\}$ for some sufficiently small r
- Let $d = \max_{x \in S} \dot{V}(x)$ [smallest neg value, slowest rate]
- This number is well-defined and negative
- $\dot{V}\big(x(t)\big) \leq d$ for all t
- $V(t) \leq V(0) + d \times t$
- But then eventually $V(t) < c$



Example: Pendulum

- $\dot{x} = x_2; \dot{x}_2 = -\text{a}\sin(x_1) - bx_2$ where $a = \frac{g}{L}$ $b = \frac{k}{m}$ equilibrium $(0{,}0)$
- Consider the Lyapunov function $V(\boldsymbol{x}) = mgL(1 - \cos x_1) + \frac{1}{2}mL^2x_2^2$
  - Observe that this is positive definite
  - Continuously differentiable
- $\dot{V}(\boldsymbol{x}) = \frac{\partial V}{\partial x} f(\boldsymbol{x}) = \sum \frac{\partial V}{\partial x_i} \dot{x}_i$

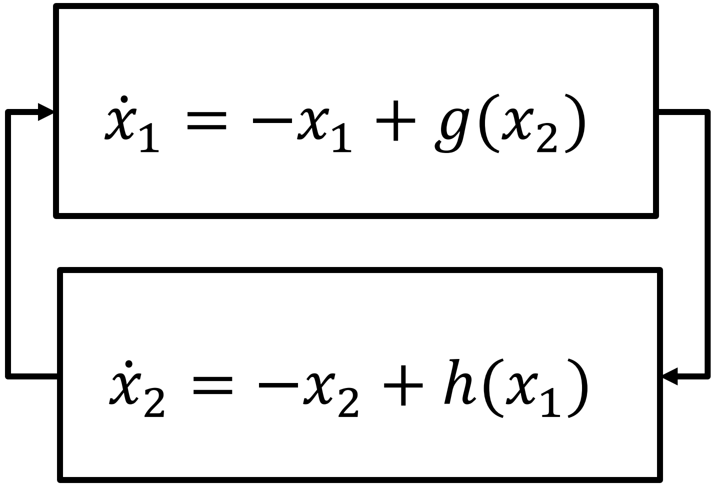
  $= mgL(\sin x_1)x_2 + mL^2x_2\left(-\frac{g}{L}\sin(x_1) - \frac{k}{m}x_2\right)$

  $= -kL^2x_2^2 \leq 0$ **Stable**
- $\dot{V}(\boldsymbol{x}) = 0$ only when $x_2 = 0$
  - The largest invariant subset of this set is $(x_1, x_2) = (0{,}0)$
  - LaSalle's invariance principle enables us to conclude **Asymptotic Stability**

Example 2: Reasoning about stability without solving ODEs

$\dot{x}_1 = -x_1 + g(x_2); \dot{x}_2 = -x_2 + h(x_1)$

Given that $|g(x_2)| \leq \frac{|x_2|}{2}, |h(x_1)| \leq \frac{|x_1|}{2}$



- Use $V = \frac{1}{2}(x_1^2 + x_2^2) \geq 0$
- $\dot{V} = x_1\,\dot{x}_1 + x_2\,\dot{x}_2$

$$= -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1)$$

$$\leq -x_1^2 - x_2^2 + \frac{1}{2}(|x_1 x_2| + |x_2 x_1|)$$

$$\leq -\frac{1}{2}(x_1^2 + x_2^2) = -V$$

$$(|x_1| - |x_2|)^2 \geq 0$$

$$x_1^2 + x_2^2 \geq 2|x_1 x_2|$$

$$|x_1 x_2| \leq \frac{1}{2}(x_1^2 + x_2^2)$$

We conclude global asymptotic stability (in fact global exponential stability) without knowing solutions

Verifying Stability for Linear Systems

Consider the linear system $\dot{x} = Ax$

Theorem.

1. It is asymptotically stable iff all the eigenvalues of *A* have **strictly** negative real parts (*Hurwitz*).

2. It is Lyapunov stable iff all the eigenvalues of *A* have real parts that are either zero or negative and the *Jordan blocks* corresponding to the eigenvalues with zero real parts are of size 1.

Jordan decomposition

For every *n x n* matrix *A*, there exists a nonsingular *n x n* matrix *P* such that

$$PAP^{-1} = J = \begin{bmatrix} J_1 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ 0 & 0 & J_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & J_\ell \end{bmatrix},$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ 0 & 0 & \lambda_i & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i \end{bmatrix}.$$

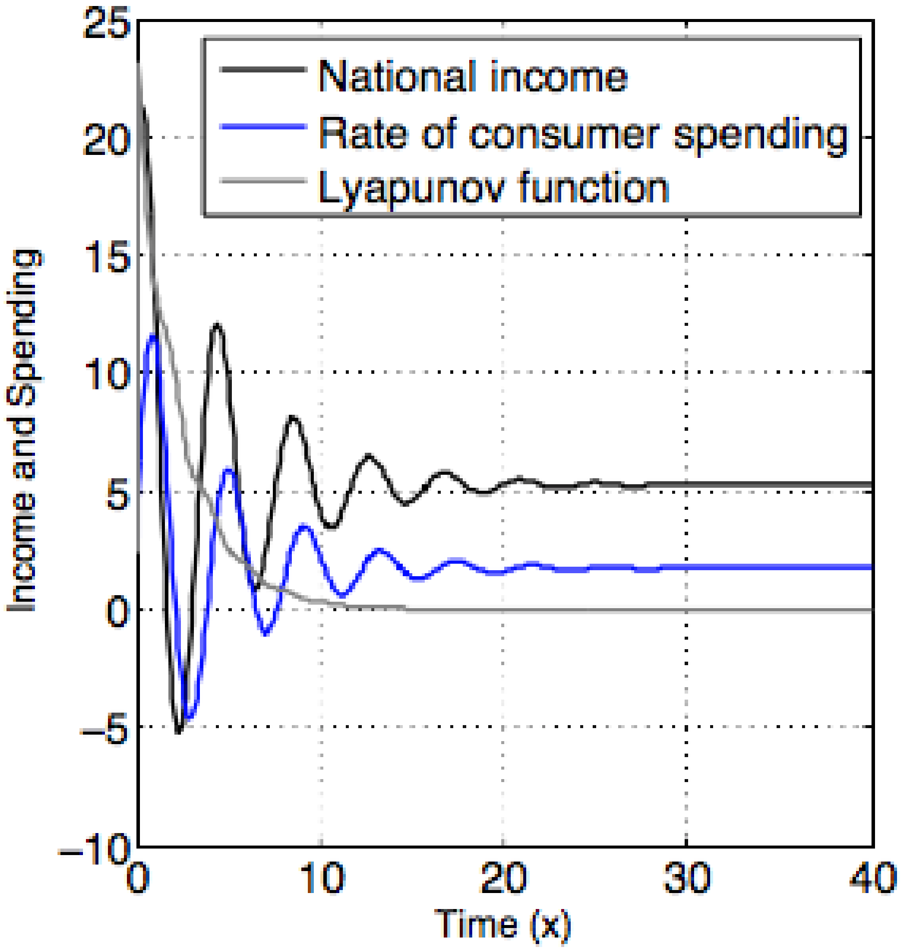
where each J_i is a upper triangular matrix called a Jordan block

Example 1: Simple model of an economy

- x: national income y: rate of consumer spending; g: rate government expenditure
- $\dot{x} = x \ - \alpha y$
- $\dot{y} = \beta(x - y - g)$
- $g = g_0 + kx$ α, β, k are positive constants
- What is the equilibrium?
- $x^* = \frac{g_0 \alpha}{\alpha - 1 - k\alpha} y^* = \frac{g_0 \alpha}{\alpha - 1 - k\alpha}$
- Dynamics:
- $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \beta(1-k) & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Example: Simple linear model of an economy

- $\alpha = 3, \beta = 1, k = 0$
- $\lambda_1, \lambda_1^* = (-.25 \pm i\ 1.714)$
- Negative real parts, therefore, asymptotically stable and the national income and consumer spending rate converge to $x = 1.764\ y = 5.294$



Proposition. If V is a Lyapunov function then every sublevel set of V is an invariant

Proof. $V\left(x(t)\right) =$

$$
\begin{aligned}
&= V\left(x(0)\right) + \int_0^t \dot{V}\left(x(\tau)\right) d\tau \\
&\leq V(x(0))
\end{aligned}
$$

An aside: Checking inductive invariants

- $\boldsymbol{A} = \langle X, Q_0, T \rangle$
  - X: set of variables
  - $Q_0 \subseteq val(X)$
  - $T \subseteq val(X) \times val(X)$ written as a program $x' \subseteq T(x)$
- How do we check that $I \subseteq val(X)$ is an inductive invariant?
  - $Q_0 \Rightarrow I(X)$
  - $I(X) \Rightarrow I(T(X))$
- Implies that $Reach_{\boldsymbol{A}}(Q_0) \subseteq I$ without computing the executions or reachable states of $\boldsymbol{A}$
- The key is to find such I

Finding Lyapunov Functions

- The key to using Lyapunov theory is to *find* a Lyapunov function and verify that it has the properties
- In general, for nonlinear systems this is hard
- There are several approaches
  – Quadratic Lyapunov functions for linear systems
  – Decide the form/template of the function (e.g., quadratic, polynomial), parameterized by some parameters and find values of the parameters so that the conditions hold (Chapter 3 last section)
  – $V(x)$ can be represented by a neural network; compute $\dot{V}(x)$ using symbolic or automatic differentiation tools, and check $\dot{V}(x) < 0, \forall x \in D$ for some bounded domain D

Linear autonomous systems

- $\dot{x} = Ax, A \in \mathbb{R}^{n \times n}$
- The Lyapunov equation: $A^T P + PA + Q = 0$

  where $P, Q \in \mathbb{R}^{n \times n}$ are symmetric

- Interpretation: $V(x) = x^T P x$ then

  $$\dot{V}(x) = (Ax)^T P x + x^T P (Ax)$$

  [using chain rule $\frac{\partial u^T P v}{\partial t} = \frac{\partial u}{\partial t} P v + \frac{\partial v}{\partial t} P^T u$]

  $$= x^T (A^T P + PA) x = -x^T Q x$$

- If $x^T P x$ is the generalized energy then $-x^T Q x$ is the associated dissipation
- Choose Q (often an identity matrix) and find P by solving Linear Matrix Inequality(LMI) feasibility problem

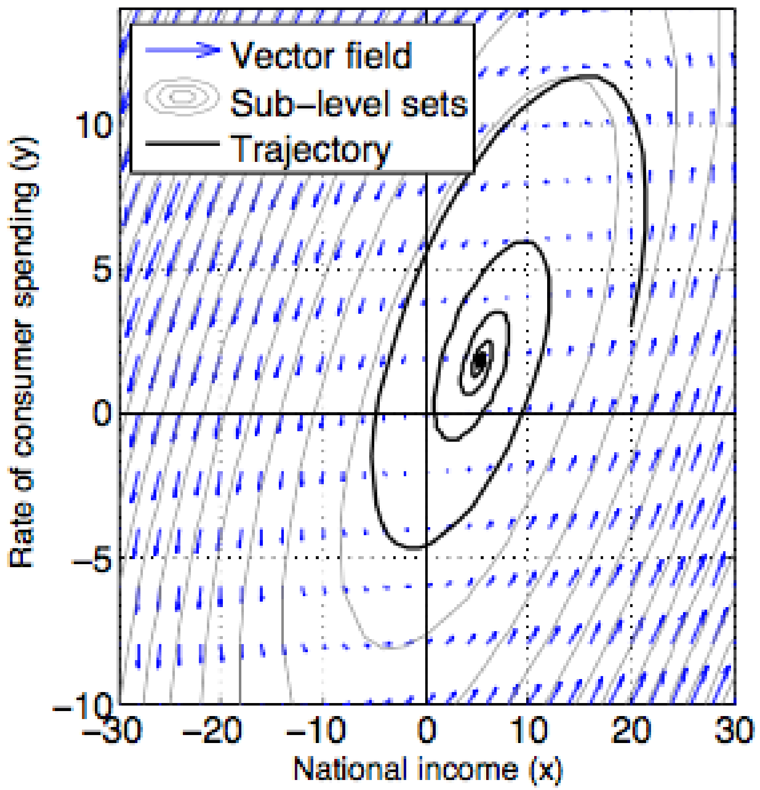
Quadratic Lyapunov Functions

- If $P > 0$ (positive definite)
- $V(x) = x^T P x = 0 \Leftrightarrow x = 0$
- The sub-level sets are ellipsoids
- If $Q > 0$ then the system is globally asymptotically stable

Same example

Lyapunov equations are solved as a set of $\frac{n(n+1)}{2}$ equations in $n(n+1)/2$ variables. Cost O(n^6)

Choose $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ solving Lyapunov equations we get $P = \begin{bmatrix} 2.59 & -2.29 \\ -2.29 & 4.92 \end{bmatrix}$ and we get the quadratic Lyapunov function $(x - x^*)P(x - x^*)^T$ an a sequence of invariants



Summary

- Key requirements for dynamical systems: stability, asymptotic stability
- For linear systems, Hurwitz conditions or LMIs for verifying stability
- For nonlinear systems, Lyapunov functions provide a general method and certificate for proving stability
- Checking Lyapunov conditions does not involve finding solutions of the system
- Finding Lyapunov functions involves solving optimization problems