Verifying Dynamical Systems: Stability

Sayan Mitra
Verifying cyberphysical systems
mitras@illinois.edu

Plan

- Dynamical system models
  - Stability
  - Lyapunov method
  - Stability of linear systems

Review dynamical systems

Physical processes are modeled by ordinary differential equations (ODEs) also known as dynamical systems

ODEs describe how variables change with time and inputs

$\frac{dx(t)}{dt} = f\big(x(t), u(t)\big)$ (1) where time $t \in \mathbb{R}$; state $x(t) \in \mathbb{R}^n$; $input\ \ u(t) \in \mathbb{R}^m$; $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

In short this is written as $\dot{x} = f(x, u)$

Given an ODE (1) and initial state $x_0 \in \mathbb{R}^n$ and input $u: \mathbb{R} \to \mathbb{R}^m$, a behavior of the system is a state trajectory or *solution* $x: \mathbb{R} \quad \to \mathbb{R}^n$.

Theorem. If $f(x(t), u(t))$ is Lipschitz continuous in the first argument and $u(t)$ is piece-wise continuous then (1) has unique solutions.

Review. Linear dynamical system

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$

Linear time invariant (LTI) system $\dot{x}(t) = Ax(t) + Bu(t)$

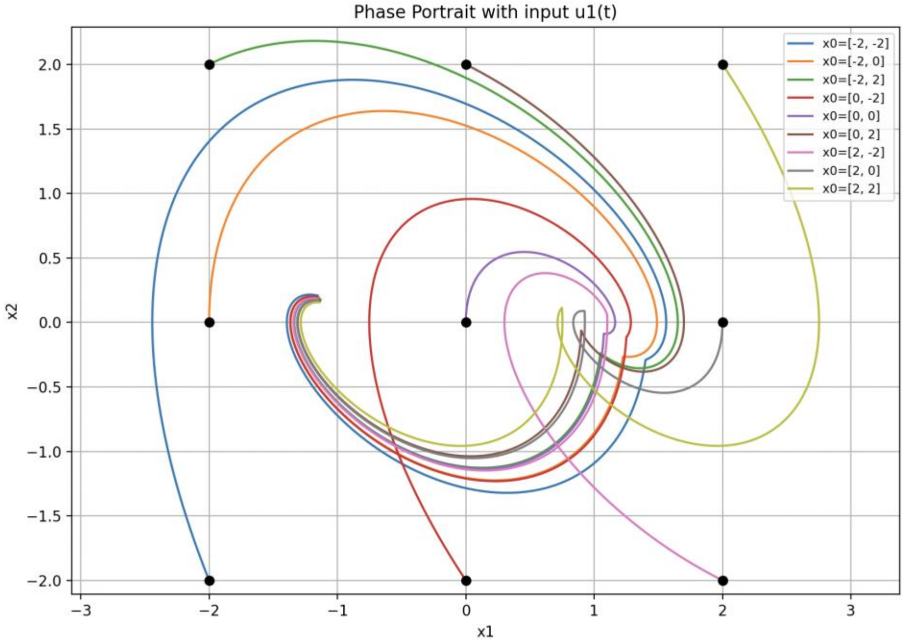
Theorem. Solutions of LTI system:

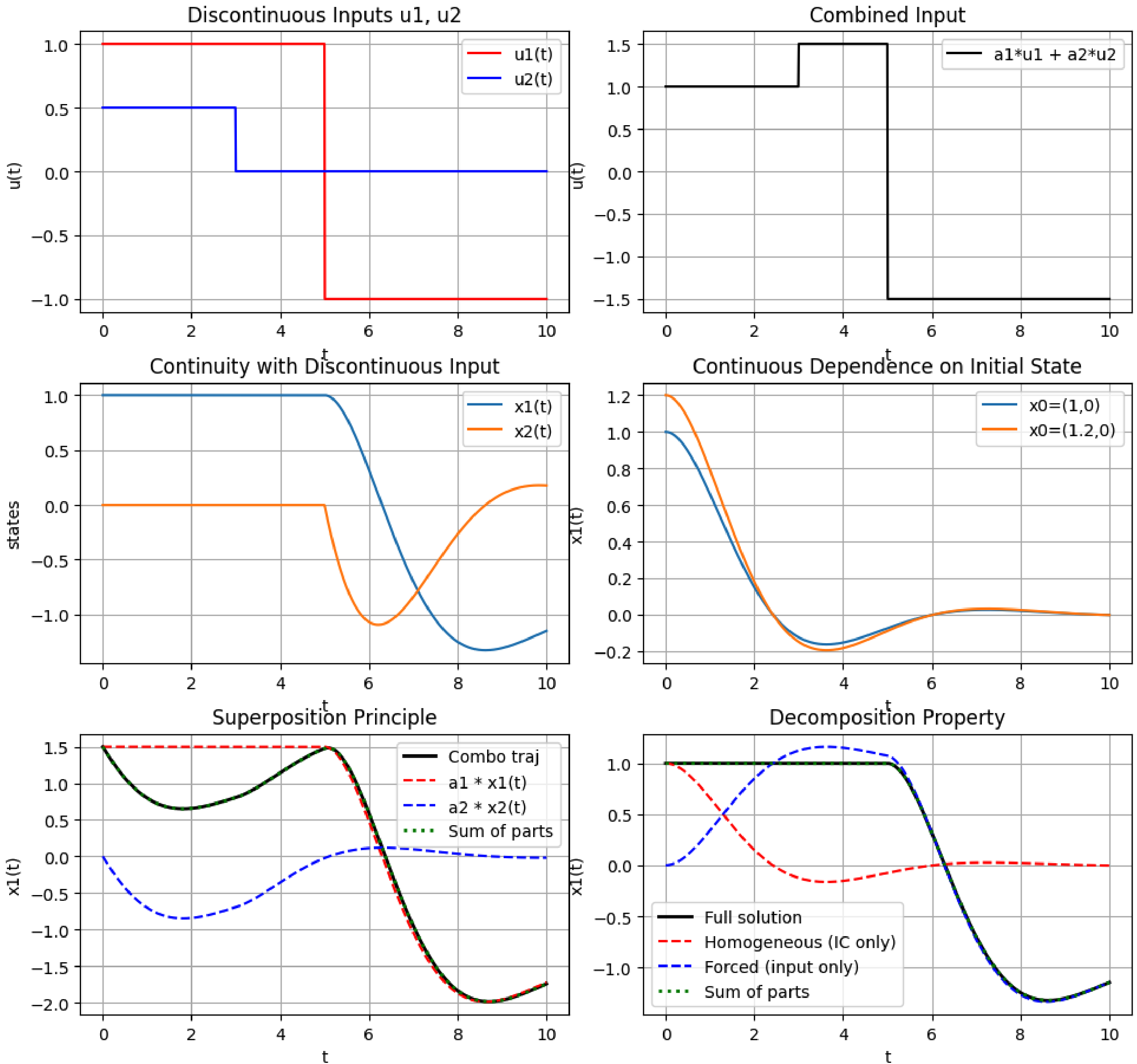
$$x(t, x_0, u) = x_0 \mathrm{e}^{At} + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} Bu(\tau) d\tau$$

Properties of linear dynamical systems

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

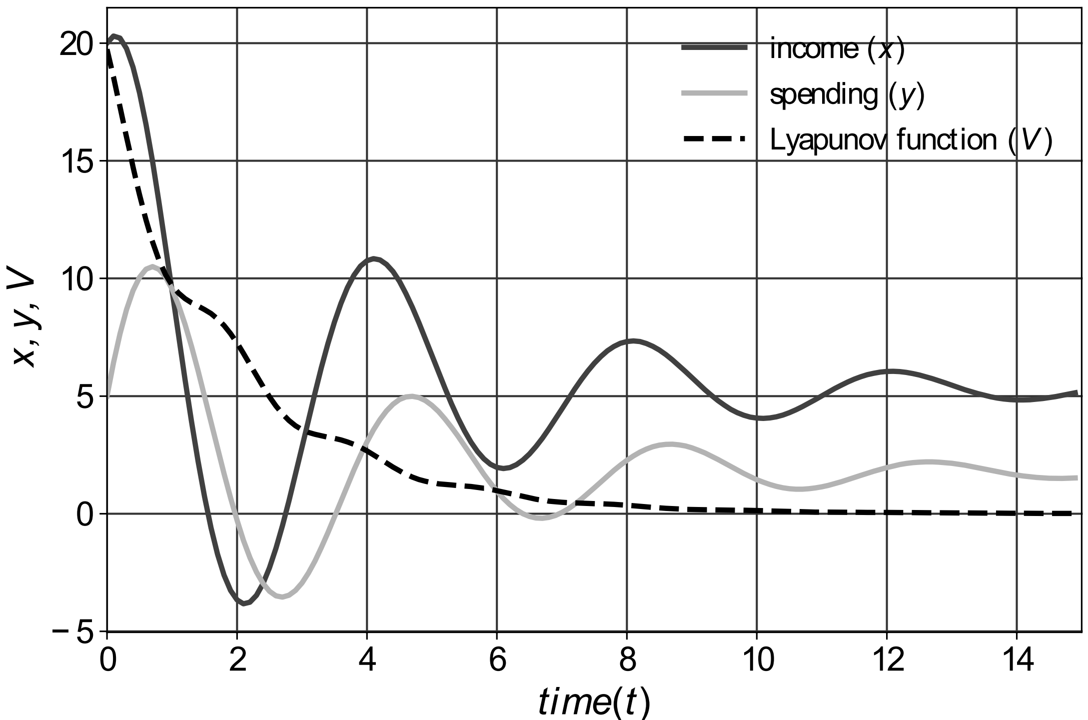




Example: Simple model of an economy

- x: national income
- y: rate of consumer spending
- g: rate government expenditure
- α: propensity to consume
- β: responsiveness of consumption

- $\dot{x} = x - \alpha y$
- $\dot{y} = \beta(x - y - g)$



Example: Gradient flow

- Consider the regression problem of fitting a straight line to data using gradient flow. The behavior of this learning algorithm can be modeled and analyzed as a dynamical system
- Given data: $\{(x_i, y_i)\}_{i=1}^{n}$ we want to fit a linear model $\hat{y}_i = \theta x_i$ by finding the ***best*** θ
- Minimizing the squared loss: $L(\theta) = \frac{1}{2}\sum_i (\theta x_i - y_i)^2$
- Gradient of loss $\nabla_\theta L(\theta) = \frac{1}{2}.2\sum_i (\theta x_i - y_i)x_i = \sum_i (\theta x_i - y_i)x_i \ = \theta \sum_i x_i^2 \ - \sum x_i y_i = \theta A - B$
- Gradient descent $\theta_{k+1} = \theta_k - \eta \nabla_\theta L(\theta_k)$, η is a parameter called learning rate
  $= \theta_k - \eta(\mathrm{A}\theta_k - B)$
- Gradient flow is the continuous-time limit

$$\frac{d\theta(t)}{dt} = -\nabla_\theta L(\theta(t))$$

$$= -A\theta + B$$ LTI system

- Solution $\theta(t) = \theta^* + (\theta(0) - \theta^*)e^{-At}$ $\theta^* = \frac{B}{A}$

$f: \mathbb{R}^n \to \mathbb{R}$
Gradient $\nabla_x f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

Example: $f(x, y) = x^2 + y^2$
$\nabla_x f = (2x, 2y)$

Gradient flow convergence to an optimal model

Consider the regression problem of fitting a straight line to data using gradient flow. The behavior of this learning algorithm can be modeled and analyzed as a dynamical system

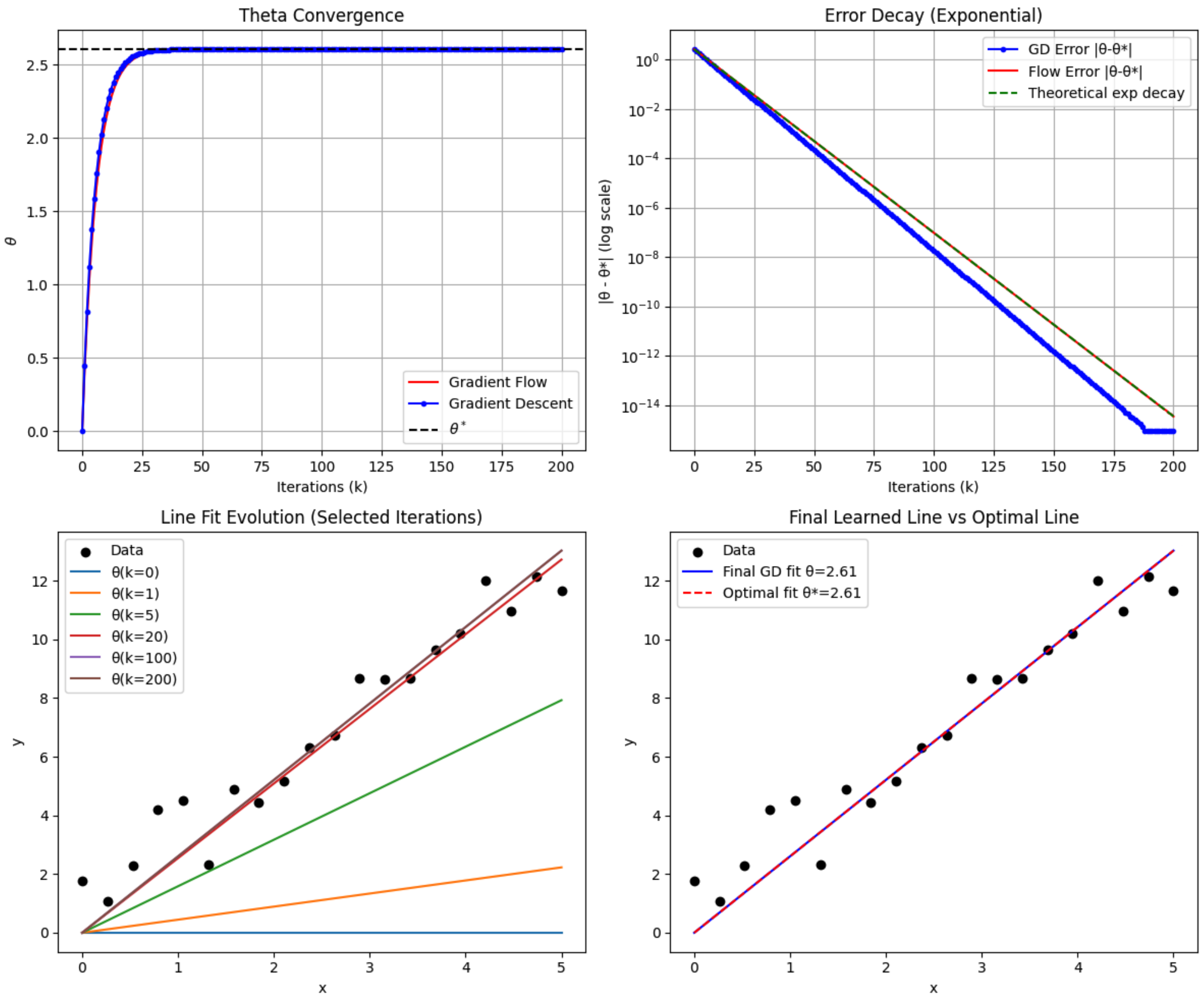
Given data: $\{(x_i, y_i)\}_{i=1}^{n}$ we want to fit a linear model $\hat{y}_i = \theta x_i$ by finding the ***best*** θ

Minimizing the squared loss: $L(\theta) = \frac{1}{2}\sum_i (\theta x_i - y_i)^2$

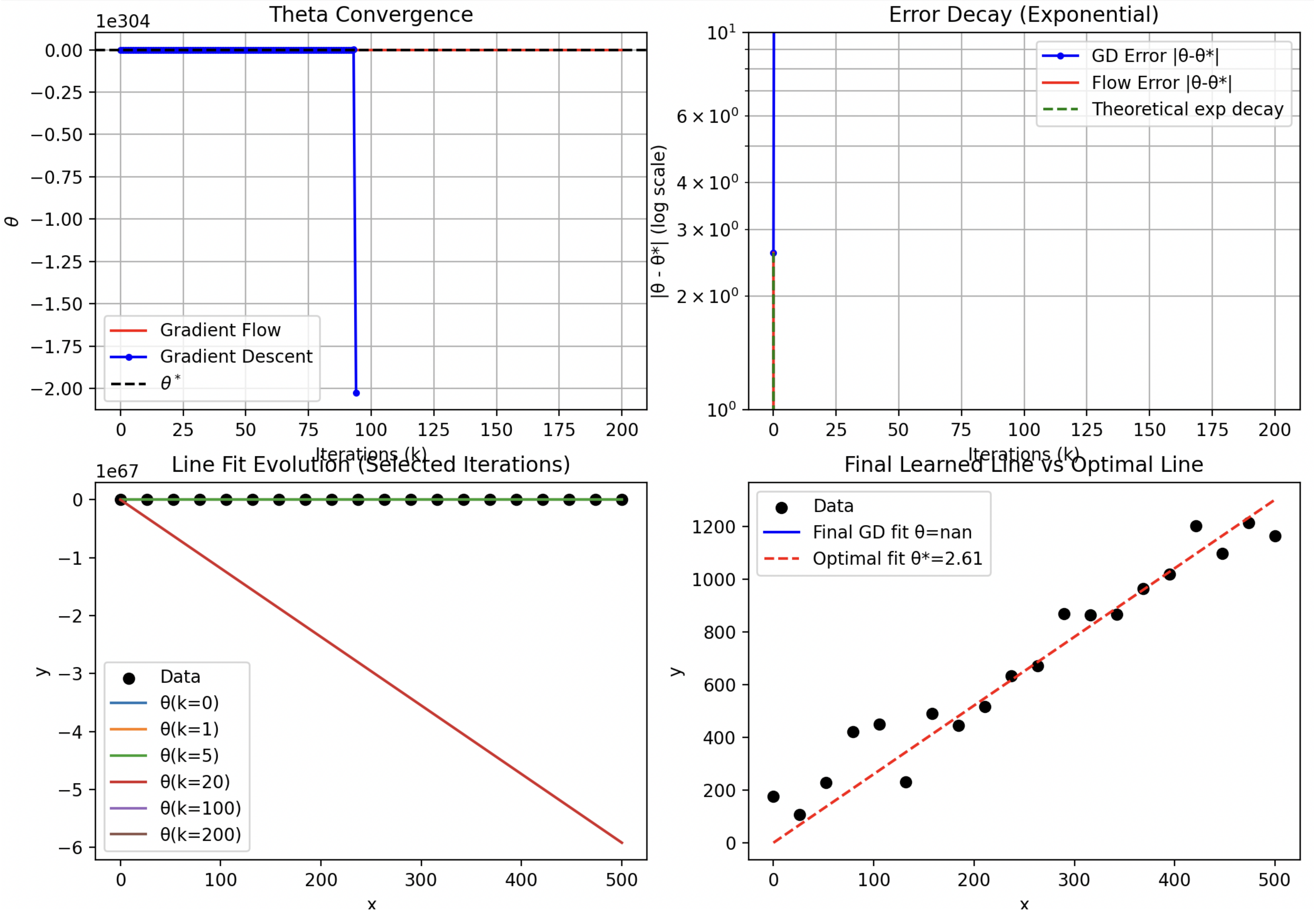
Gradient of loss $\nabla_\theta L(\theta) = \theta A - B$

Gradient descent $\theta_{k+1} = \theta_k - \eta(A\theta_k - B)$

Gradient flow $\frac{d\theta(t)}{dt} = -A\theta + B$



Grad flow converging but to an undesirable value



Requirements for dynamical systems

What type of properties are we interested in?

- Invariance (as in the case of automata)
- State remains bounded
- **State converges to target**
- Bounded input or disturbance gives bounded output (BIBO)

Requirements: Stability

- We will focus on time invariant autonomous systems (closed systems, systems without inputs)
- $\dot{x}(t) = f\big(x(t)\big), x_0 \in \mathbb{R}^n, t_0 = 0$ *–(1)*
- $x(t)$ is the solution
- $|x(t)|$ norm
- $x^* \in \mathbb{R}^n$ is an **equilibrium point** if $f(x^*) = 0$.
- For analysis we will assume $x^* = \mathbf{0}$ to be an equilibrium point of (1) with out loss of generality

Example: Pendulum

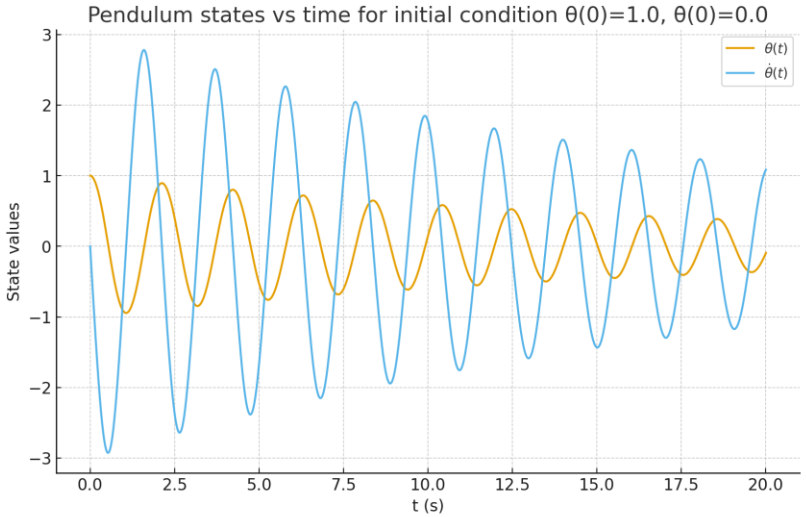
Pendulum equation

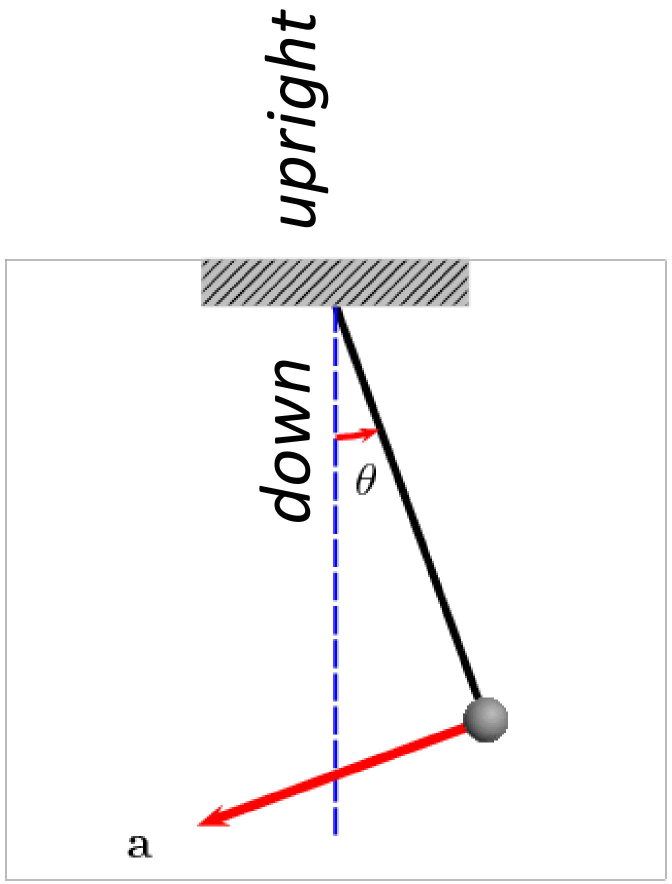
$x_1 = \theta \quad x_2 = \dot{\theta}$

$x_2 = \dot{x}_1$

$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

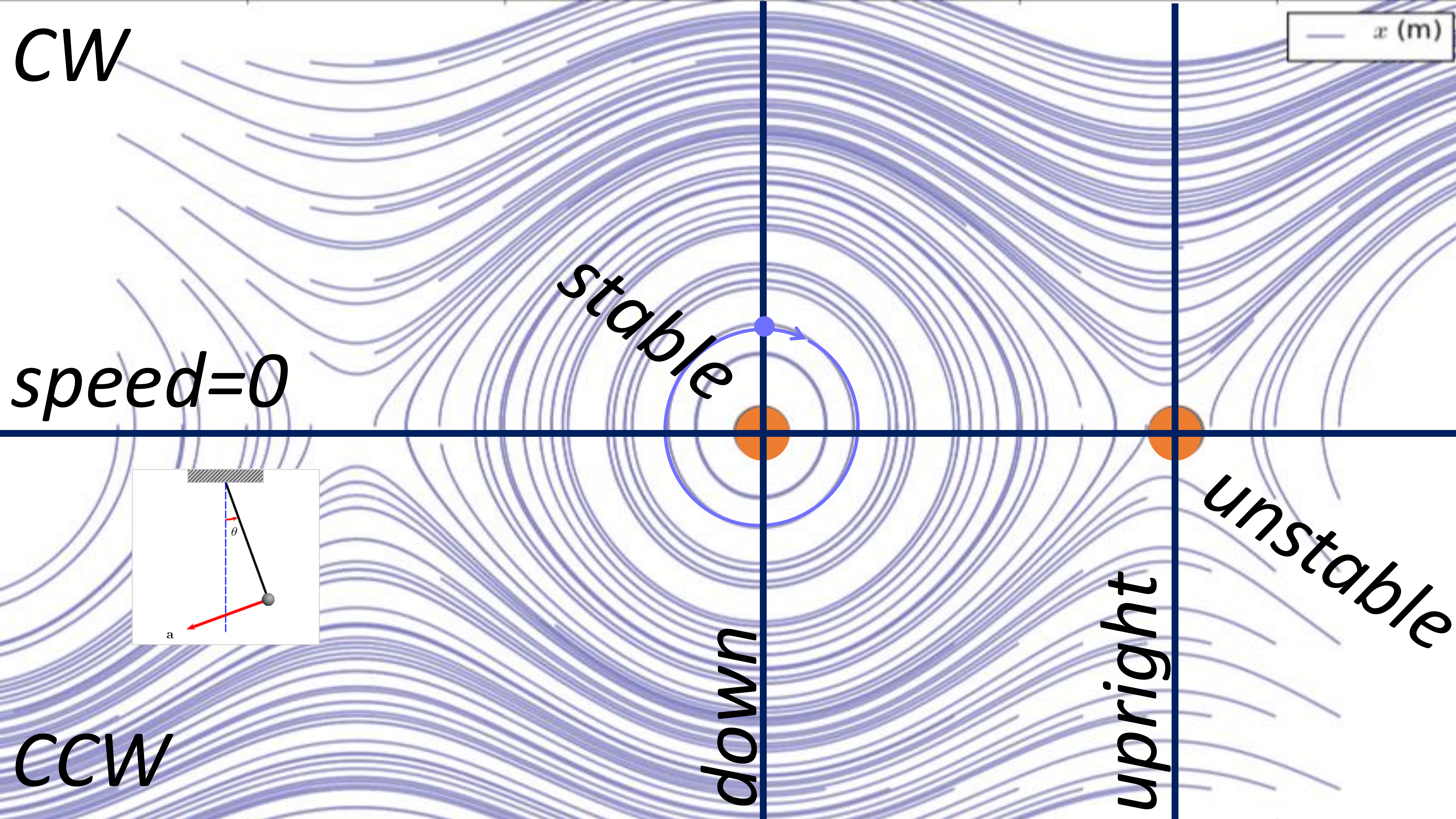
$$\begin{bmatrix}\dot{x}_2 \\ \dot{x}_1\end{bmatrix} = \begin{bmatrix}-\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \\ x_2\end{bmatrix}$$





k: friction coefficient

Two equilibrium points: $(0,0), (\pi, 0)$


CW
x (m)
stable
speed=0
unstable
θ
a
down
upright
CCW

Aleksandr M. Lyapunov

Aleksandr Mikhailovich Lyapunov (June 6 1857–November 3, 1918), Russian mathematician and physicist.

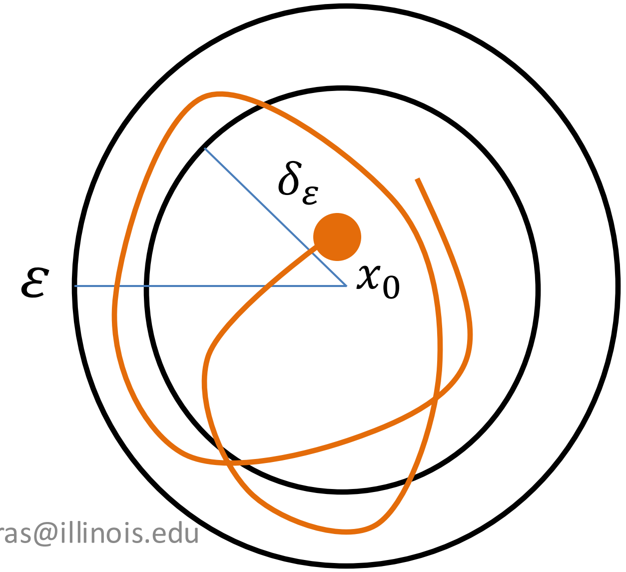
Lyapunov stability

Lyapunov stability: The system (1) is said to be ***Lyapunov stable*** (at the origin) if

$\forall \varepsilon > 0 \ \exists \ \delta_\varepsilon > 0$ such that

$$|x_0| \leq \delta_\varepsilon \Rightarrow \forall \ \mathrm{t} \geq 0, |\xi(x_0, t)| \leq \varepsilon.$$

How is this related to invariants and reachable states ?

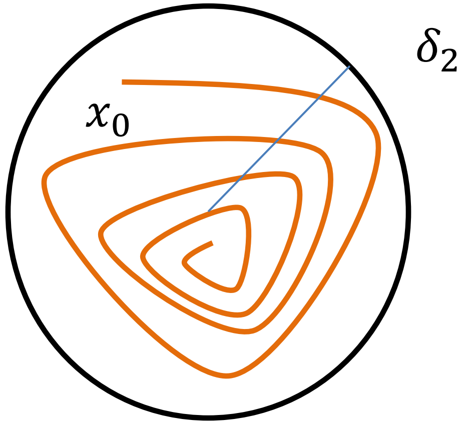


Asymptotically stability

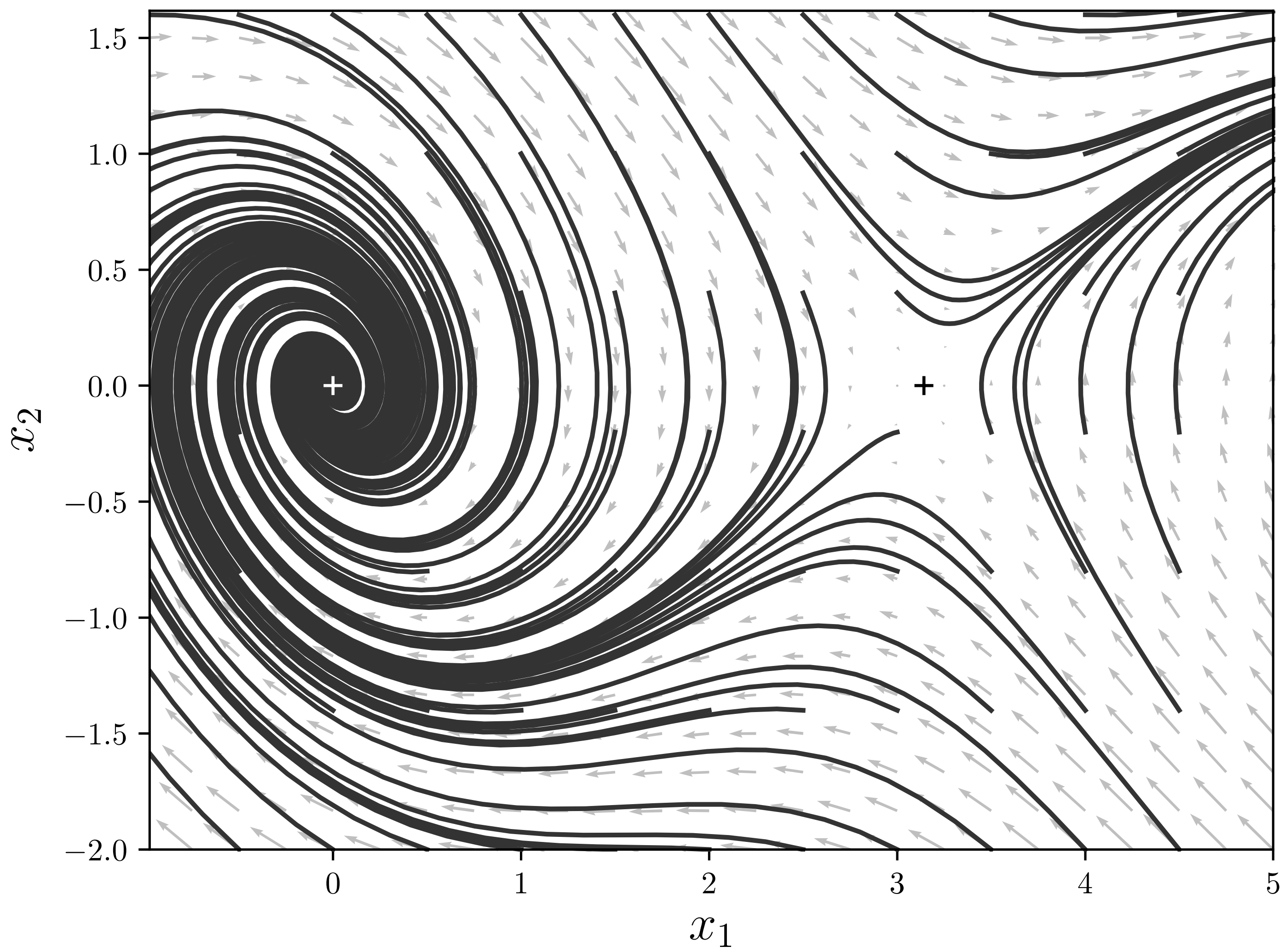
The system (1) is said to be ***Asymptotically stable*** *(at the origin)* if it is Lyapunov stable and

$\exists \delta_2 > 0$ such that $\forall |x_0| \leq \delta_2$ as $t \rightarrow \infty, |\xi(x_0, t)| \rightarrow \mathbf{0}$.

If the property holds for any δ_2 then **Globally Asymptotically Stable**



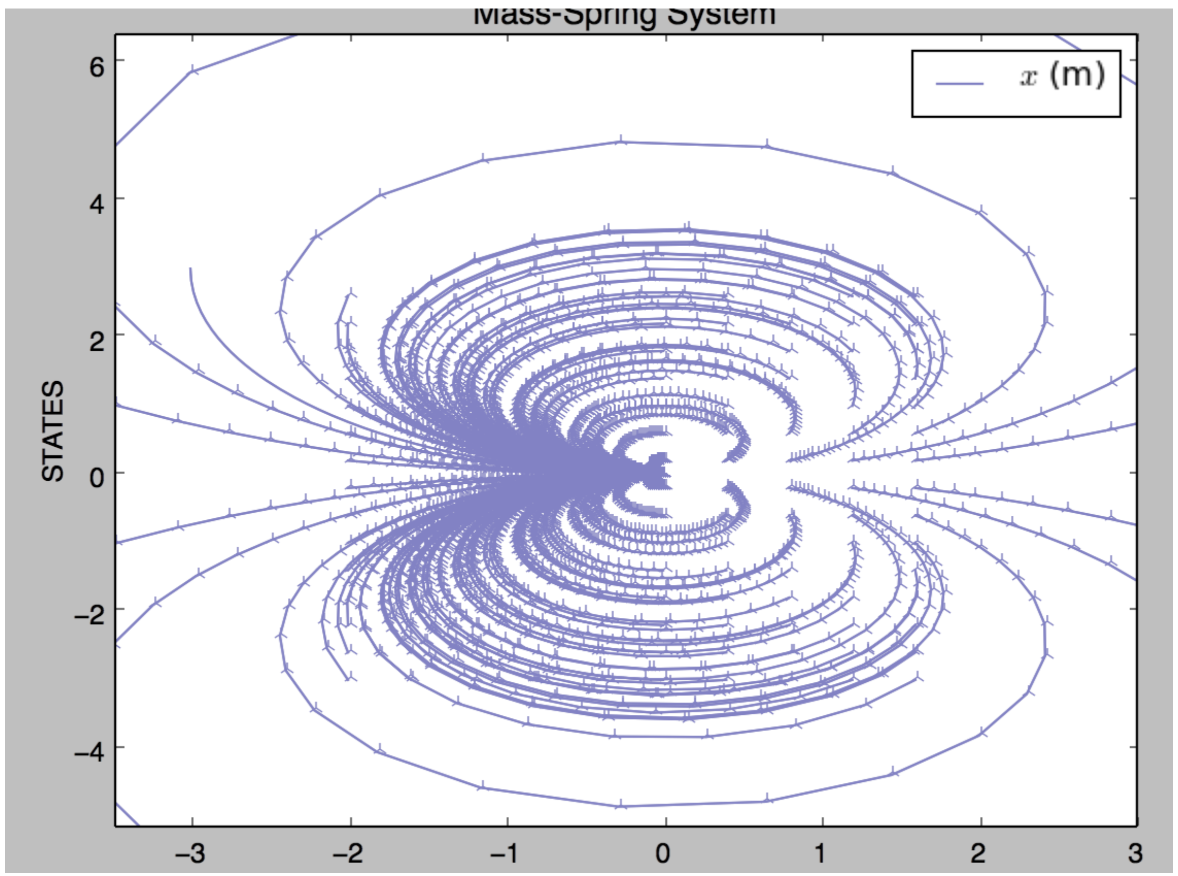
Phase portrait of pendulum with friction



Butterfly*

$$\begin{bmatrix}\dot{x_2}\\ \dot{x_1}\end{bmatrix}=\begin{bmatrix}2x_1x_2\\ x_1^2-x_2^2\end{bmatrix}$$

All solutions converge to 0 but the equilibrium point (0,0) is not Lyapunov stable



*Not discussed in class

Van der pol oscillator

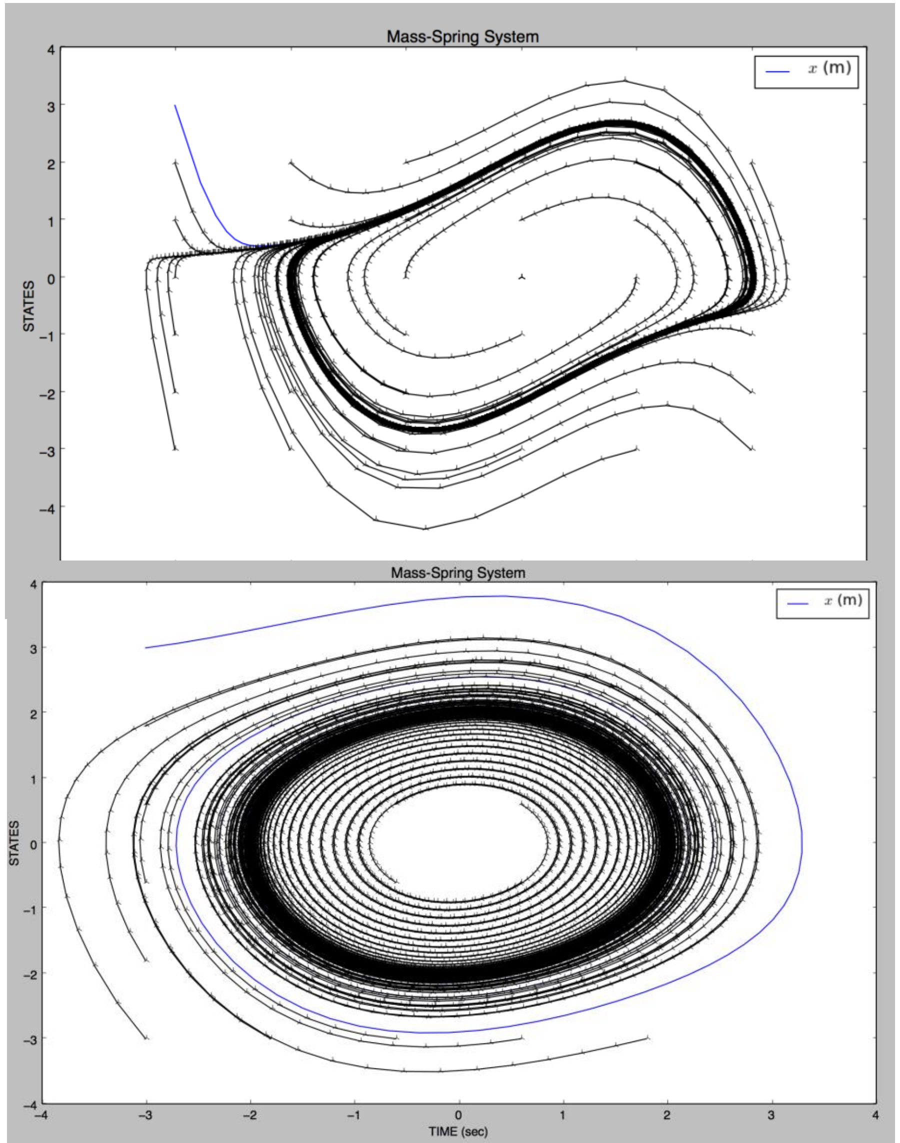
Van der pol oscillator

$$\frac{dx^2}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

$x_1 = x; x_2 = \dot{x}_1;$
coupling coefficient $\mu = 2 \;\; 0.1$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} \mu(1 - x_1^2)x_2 - \; x_1 \\ x_2 \end{bmatrix}$$

stable ?



Stability of solutions* (instead of points)

- For any $\xi \in \mathrm{PC}(\mathbb{R}^{\geq 0}, \mathbb{R}^n)$ define the s-norm $\left||\xi|\right|_s = \sup_{t\in\mathbb{R}} ||\xi(t)||$

- A dynamical system can be seen as an operator that maps initial states to signals $T: \mathbb{R}^n \to PC(\mathbb{R}^{\geq 0}, \mathbb{R}^n)$

- Lyapunov stability required that this operator is continuous

- The solution ξ^* is ***Lyapunov stable*** if T is continuous as $\xi^*(0)$. $\mathrm{i.e.}$, for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every $x_0 \in \mathbb{R}^n$ if $|\xi^*(0) - x_0| \leq \delta_\varepsilon$ then $\left|\left|T\left(\xi^*(t)\right) - T(x_0)\right|\right|_s \leq \varepsilon$.

*Not discussed in class

Verifying Termination

How to verify that a loop terminates?

While $(x > 0)$

$x = f(x)$, where $f: \mathbb{Z}^n \to \mathbb{Z}^n$

Come up with a **ranking function** $V: \mathbb{Z}^n \to \mathbb{N}$ such that $V\big(f(x)\big) < V(x)$

Input $x_1 \geq x_2 > 0$

While $(x_2 > 0)$

$x_1, x_2 = x_2, x_1 \; mod \; x_2$

Claim. $V(x_1, x_2) = x_1 + x_2$ works

$V(x_1, x_2) \in \mathbb{N}$

$V(f(x_1, x_2))$

$= x_2 + \; x_1 mod \; x_2$

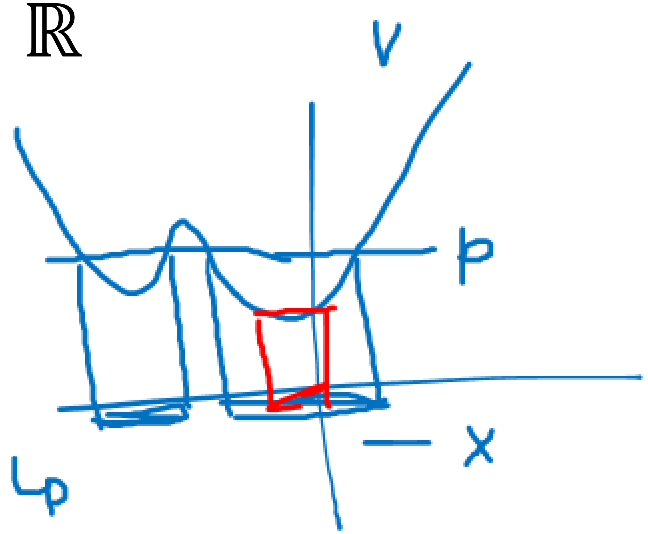
$< x_2 + x_2$ [$x_1 mod \; x_2 < x_2$ as loop stops $x_2 = 0$]

$< 2x_2 \leq x_1 + x_2 = V(x_1, x_2)$

Last inequality uses the invariant: $x_1 \geq x_2$

Stability of nonlinear systems

- For any ***positive definite*** function of state $V: \mathbb{R}^n \to \mathbb{R}$
  - $V(x) \geq 0$ and $V(x) = 0$ iff $x = 0$
- ***Sublevel sets*** of $L_p = \{x \in \mathbb{R}^n \mid V(x) \leq p\}$
- $V(x(t))$



V differentiable with continuous first derivative

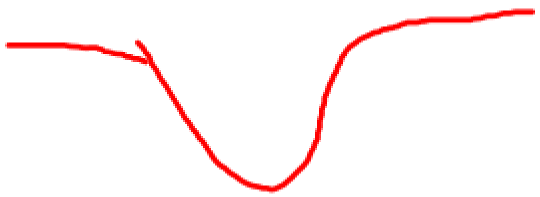
- $\dot{V} = d\frac{V(x(t))}{dt} = ?$
- $\frac{\partial V}{\partial x} . \frac{d}{dt}(x(t)) = \frac{\partial V}{\partial x} . f(x)$ is also continuous
- V is **radially unbounded** if $||x|| \to \infty \Rightarrow V(x) \to \infty$

Verifying Stability

Theorem. (Lyapunov) Consider the system (1) with state space $x(t) \in \mathbb{R}^n$ and suppose there exists a **positive definite, continuously differentiable** function $V: \mathbb{R}^n \to \mathbb{R}$. The system is:

1. Lyapunov stable if $\dot{V}\big(x(t)\big) = \frac{\partial V}{\partial x} f(x) \leq 0$, for all $x \neq 0$

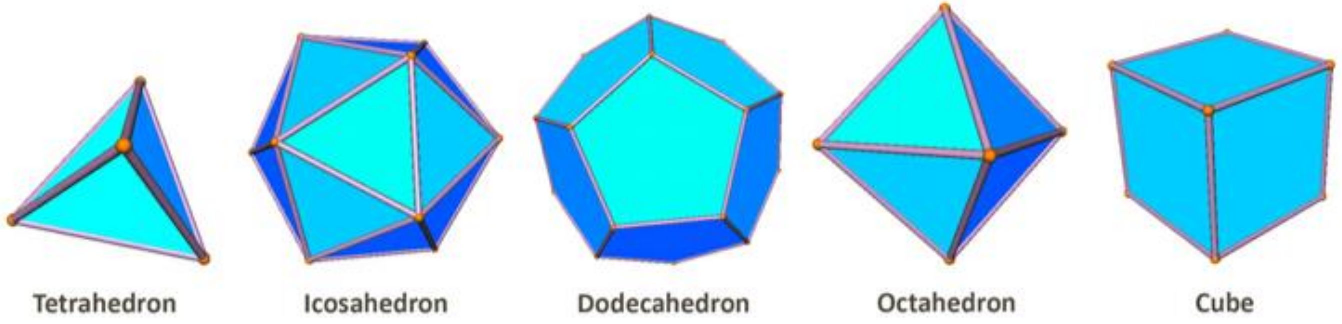
2. Asymptotically stable if $\dot{V}\big(x(t)\big) < 0$, for all $x \neq 0$



3. It is globally AS if V is also radially unbounded.

Small puzzle

- Platonic solids. Solid bodies whose faces are regular polygons, all identical, that meet in identical fashion at every vertex. How many such are there? Exactly five!



Platonic solids.
Wilczek, Frank. A Beautiful Question: Finding Nature's Deep Design (p. 39).