Modeling Physics: Dynamical Systems

Sayan Mitra

Verifying cyberphysical systems

mitras@illinois.edu

Plan

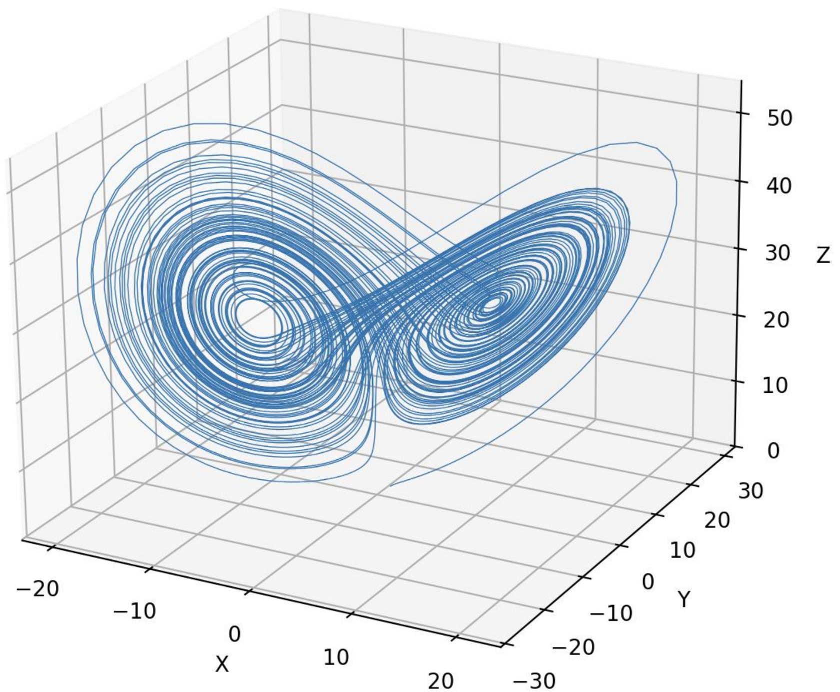
- Dynamical system models
  - notions of solutions
  - Linear dynamical systems
  - Connection to automata
  - Stability
  - Lyapunov method

Example: The Lorentz Attractor

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

$$\boldsymbol{x} = [x, y, z] \in \mathbb{R}^3$$

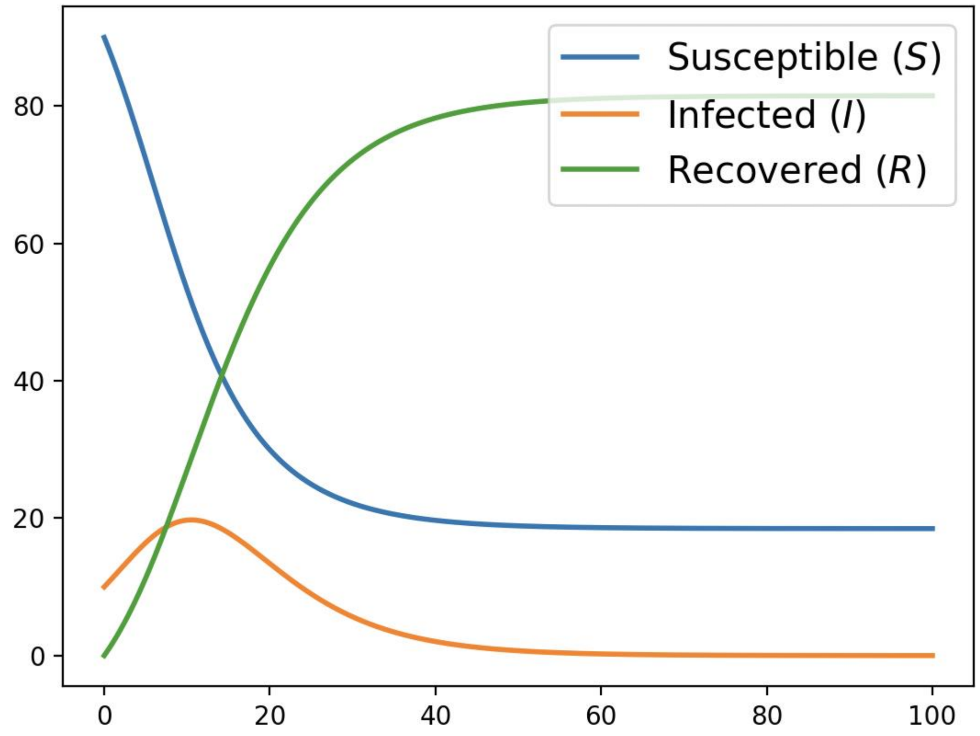
$$\begin{bmatrix} \dot{x} \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sigma(y - x) \\ x(\rho - z) - y \\ xy - \beta z \end{bmatrix}$$



Example: Epidemiology

- $S(t)$: number of **susceptible** individuals
- $I(t)$: number of **infected** individuals
- $R(t)$: number of **recovered** (or removed) individuals
- β: infection/transmission rate
- γ: recovery rate (~5-7 days = 0.14-0.2 for SARS-COV-2)

$$\frac{dS}{dt} = -\frac{\beta}{N} SI$$
$$\frac{dI}{dt} = -\frac{\beta}{N} SI - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$



Introduction to dynamical systems

Behaviors of physical processes are described in terms of instantaneous laws

Common notation: $\frac{dx(t)}{dt} = f(x(t), u(t), t) \;\; - (1),$

where time $t \in \mathbb{R}$; state $x(t) \in \mathbb{R}^n$; $input \;\; u(t) \in \mathbb{R}^m$; $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n$

Example. $\frac{dx(t)}{dt} = v(t)$; $\frac{dv(t)}{dt} = -g$

Initial value problem: Given system (1) and initial state $x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}$, and input $u: \mathbb{R} \rightarrow \mathbb{R}^m$, find a state trajectory or *solution* of (1).

Notions of solution

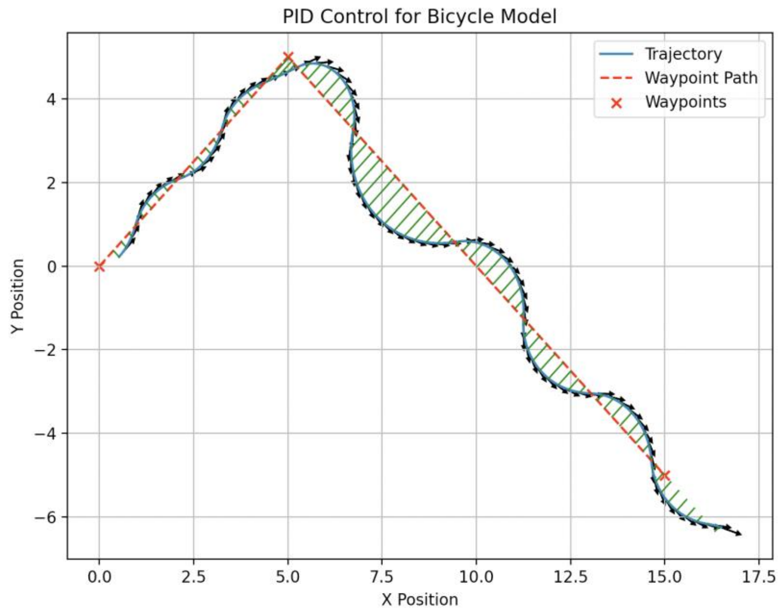
What is a solution? Many different notions.

Definition 1. (First attempt) Given x_0 and u, $x: \mathbb{R} \to \mathbb{R}^n$ is a solution or trajectory iff

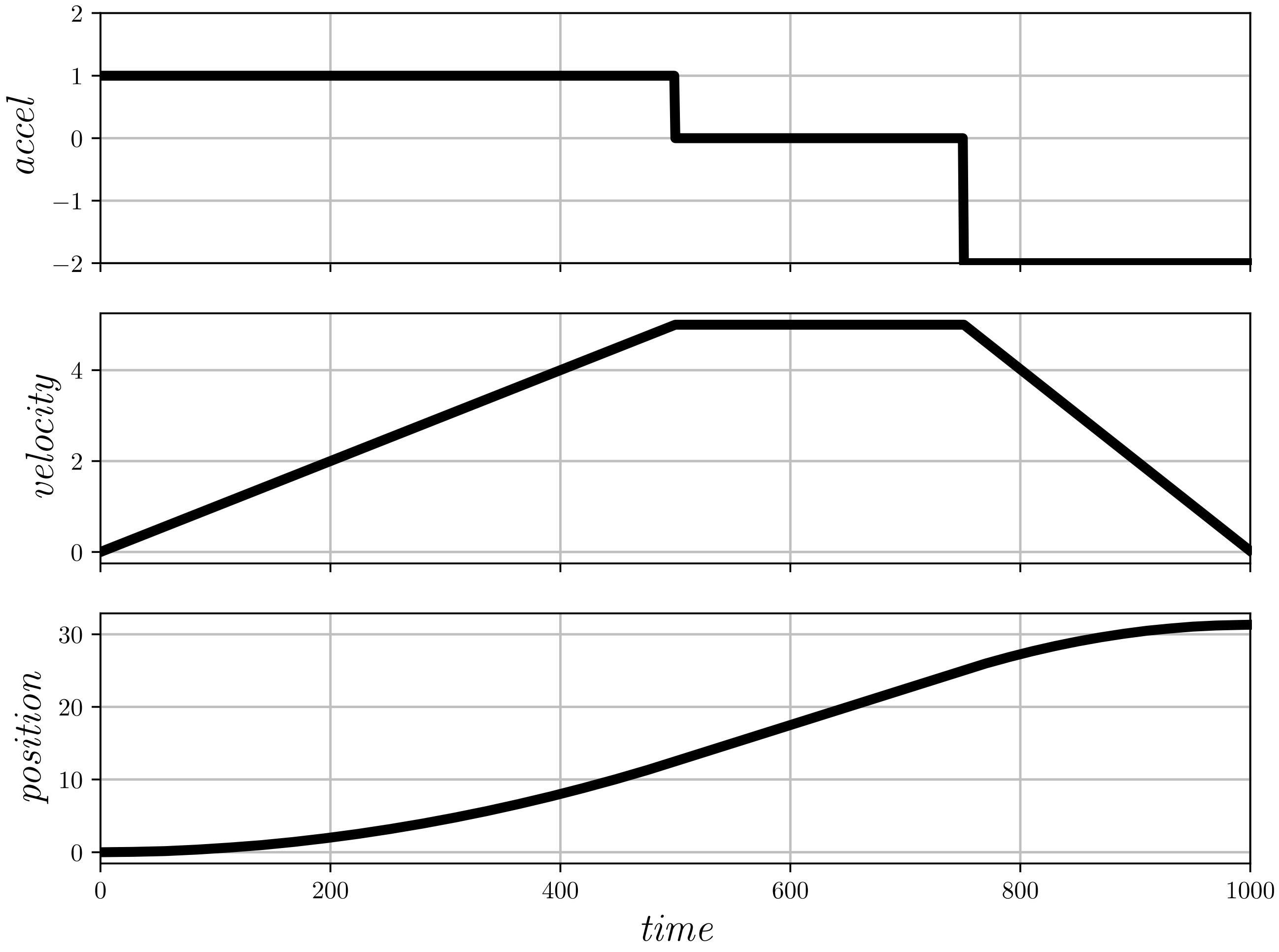
(1) $x(t_0) = x_0$ and

(2) $\frac{d}{dt}x(t) = f(x(t), u(t), t)), \forall t \in \mathbb{R}$.

Mathematically makes sense, but too restrictive. Assumes that $x(t)$ is not only continuous, but also differentiable. This disallows $\mathrm{u}(t)$ to be discontinuous, which is often required for optimal control.



Getting from point a to point b



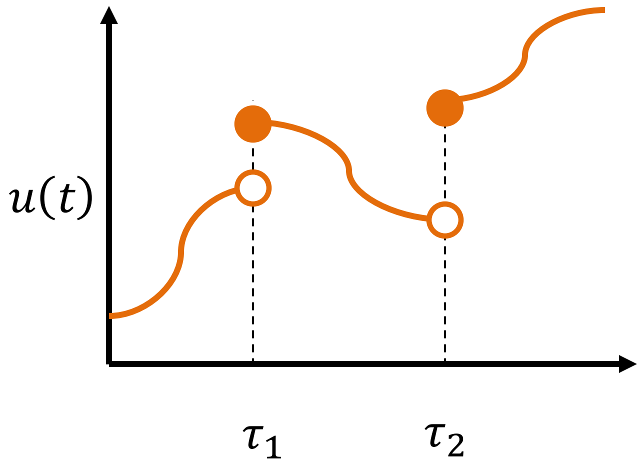
Modified notion

Definition. $u(\cdot)$ is a piece-wise continuous with set of discontinuity points $D \subseteq \mathbb{R}^m$ if

(1) $\forall \tau \in D, \lim_{t \to \tau^+} u(t) < \infty; \lim_{t \to \tau^-} u(t) < \infty$

(2) Continuous from right $\lim_{t \to \tau^+} u(t) = u(t)$

(3) $\forall\, t_0 < t_1$, $[t_0, t_1] \cap D$ is finite



$PC([t_0, t_1], \mathbb{R}^m)$ is the set of all piece-wise continuous functions over the domain $[t_0, t_1]$

Define $p(x(t), t) = f(x(t), u(t), t)$, for a given $u(t)$. Since $u(t)$ is PC in t so is p in the second argument.

Definition 2. Given x_0 and u, ξ: $\mathbb{R} \to \mathbb{R}^n$ is a solution or trajectory iff (1) $x(t_0) = x_0$ and (2) $\frac{d}{dt} x(t) = p(x(t), t), \forall t \in \mathbb{R} \backslash \mathrm{D}$.

Is PC input $u(t)$ adequate for guaranteeing existence of solutions?

Example. $\dot{x}(t) = -sgn\big(x(t)\big); x_0 = c; t_0 = 0; c > 0$
Solution: $x(t) = c - t$ for $t \le c$; check $\dot{x}(t) = -1$
Problem: f discontinuous is x

Example. $\dot{x}(t) = x^2; x_0 = c; t_0 = 0; c > 0$
Solution: $x(t) = \frac{c}{1-tc}$ works for $t < 1/c$; check $\dot{x}$
Problem: As $t \to \frac{1}{c}$ then $x(t) \to \infty$; p grows too fast

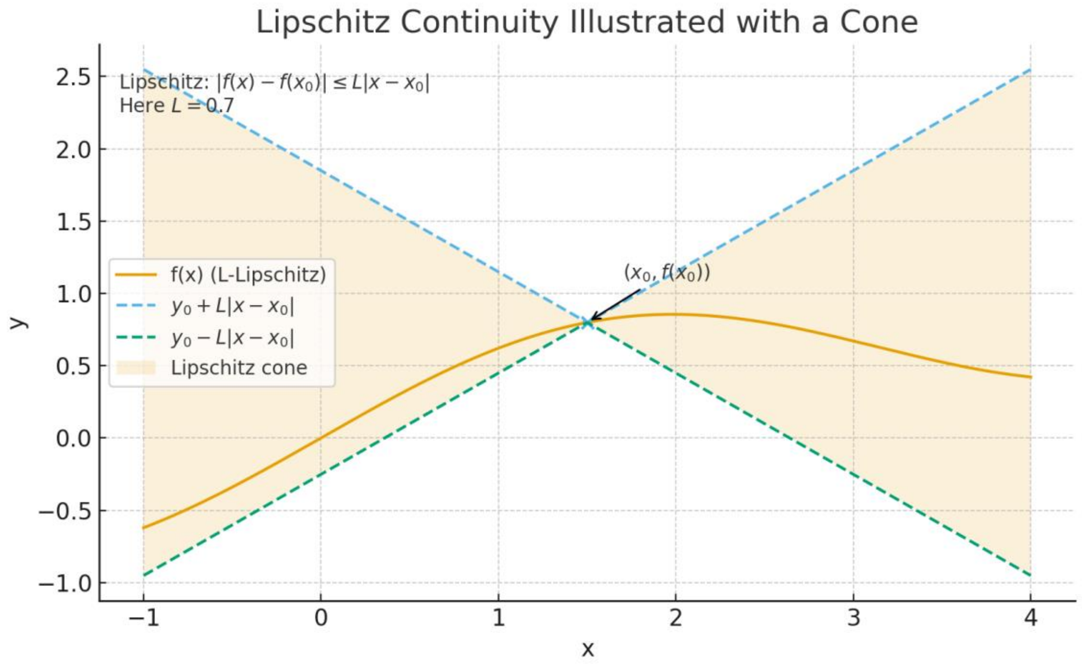
Lipschitz continuity

A function $f: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous if there exist $L > 0$ such that for any pair $x, x' \in \mathbb{R}^n$,

$$\left|\left|f(x) - f(x')\right|\right| \leq L\left|\left|x - x'\right|\right|$$

Examples: $6x + 4$; $|x|$; all differentiable functions with bounded derivatives

Non-examples: $\sqrt{x}$; x^2 (locally Lipschitz)



Existence and uniqueness of solutions

Theorem. If $p(x(t), x)$ is Lipschitz continuous in the first argument then (1) has unique solutions.

Exercise. Write an automaton / transition system model corresponding to the discretized solution of a dynamical system

Linear system and solutions

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$

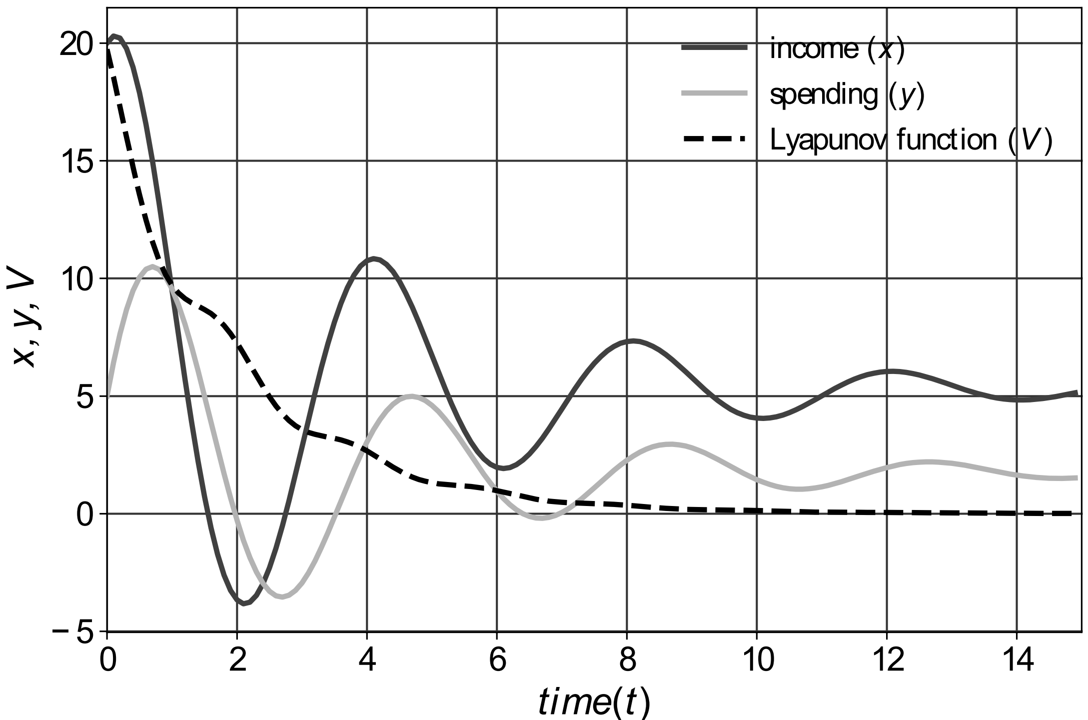
For a given initial state $x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R} and\ u(.) \in PC(\mathbb{R}, \mathbb{R}^n)$ the *solution* is a function $x(., t_0, x_0, u): \mathbb{R} \rightarrow \mathbb{R}^n$

Restrictive but important and widely-studied class of dynamical systems

Example: Simple model of an economy

- x: national income
- y: rate of consumer spending
- g: rate government expenditure
- α: propensity to consume
- β: responsiveness of consumption

- $\dot{x} = x - \alpha y$
- $\dot{y} = \beta(x - y - g)$



Linear time-varying systems

In general, for nonlinear dynamical systems we do not have closed form solutions for $x(t)$, but there are numerical solvers like CAPD, VNODE

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$ --- (2)
$y(t) = C(t)x(t) + D(t)u(t)$

$u(t)$ continuous everywhere except D_x

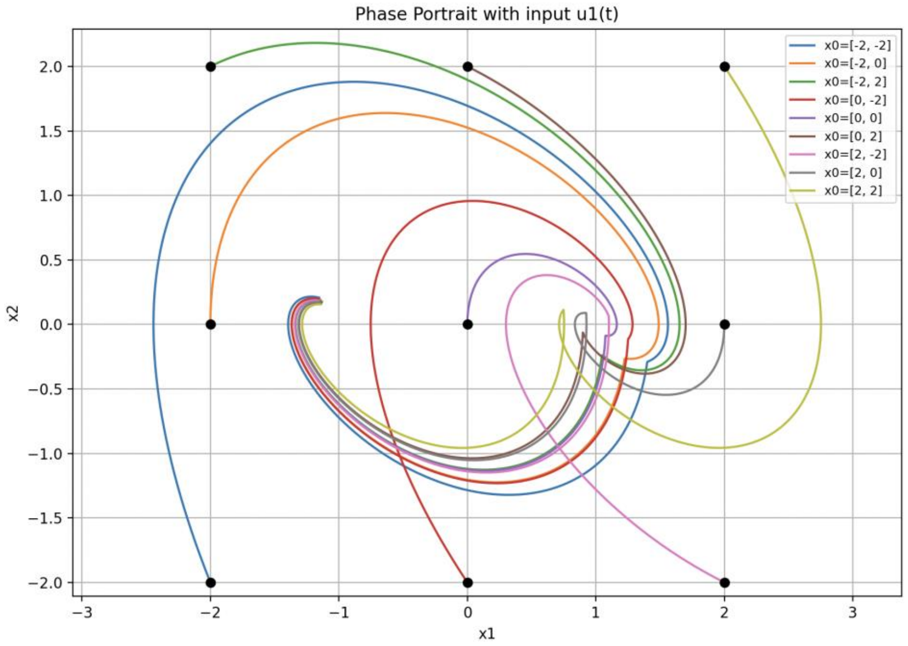
Theorem. Let $x(t, t_0, x_0, u)$ be the solution for (2) with points of discontinuity , D_x

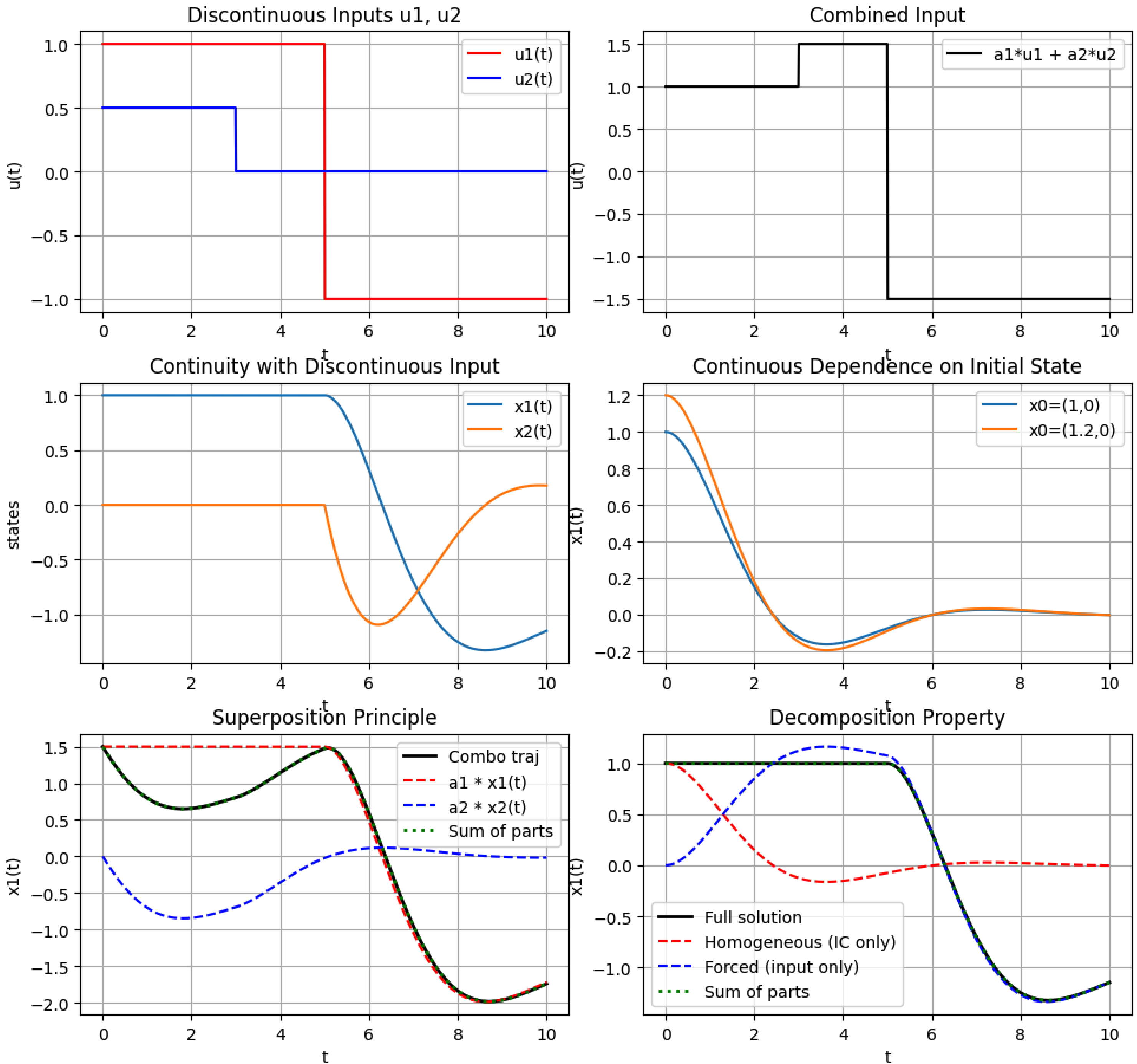
1. $\forall t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m), x(\cdot, t_0, x_0, u): \mathbb{R} \to \mathbb{R}^n$ is continuous and differentiable $\forall\ t \in \mathbb{R} \setminus D_x$
2. $\forall t, t_0 \in \mathbb{R}, u \in PC(\mathbb{R}, \mathbb{R}^m), x(t, t_0, \cdot, u): \mathbb{R}^n \to \mathbb{R}^n$ is continuous
3. $\forall t, t_0 \in \mathbb{R}, x_{01}, x_{02} \in \mathbb{R}^n, u_1, u_2 \in PC(\mathbb{R}, \mathbb{R}^m), a_1, a_2 \in \mathbb{R}, x(t, t_0, a_1 x_{01} + a_2 x_{02}, a_1 u_1 + a_2 u_2) = a_1 x(t, t_0, x_{01}, u_1) + a_2 x(t, t_0, x_{02}, u_2)$
4. $\forall t, t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m), x(t, t_0, x_0, u) = x(t, t_0, x_0, \mathbf{0}) + x(t, t_0, 0, u)$

Properties of linear dynamical systems

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$





Linear system and solutions

- Since $x(.,t_0,x_0,u)$: $\mathbb{R} \to \mathbb{R}^n$ is a linear function of the initial state and input,
- $x(t,t_0,x_0,u)$ = $x(t,t_0,0,u)$ + $x(.,t_0,x_0,0)$
- Let us focus on the linear function $x(.,t_0,x_0,0)$
- Define $\Phi(.,t_0)x_0 = x(.,t_0,x_0,u)$
- This $\Phi(.,t_0)$:$\mathbb{R} \to \mathbb{R}^{n\times n}$ is called the state transition matrix

Properties of Φ

- $\Phi(.,t_0): \mathbb{R} \rightarrow \mathbb{R}^{n\times \mathrm{n}}$ is the unique solution of (2) and is defined by a (Peano-Baker) infinite sequence of integrals
- $\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0)$ with $\Phi(t,t) = I$
  - Continuous everywhere
  - Differentiable everywhere except D_x ($A(t)$ isn't)
- $\forall t_0, t_1, t\ \Phi(t,t_0) = \Phi(t,t_1)\Phi(t_1,t_0)$
- $\Phi(t,t_0)$ is invertible $[\Phi(t,t_0)]^{-1} = \Phi(t_0,t)$

Solution of linear systems in Φ

Theorem.

$$x(t, t_0, x_0, u) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

Linear time invariant system

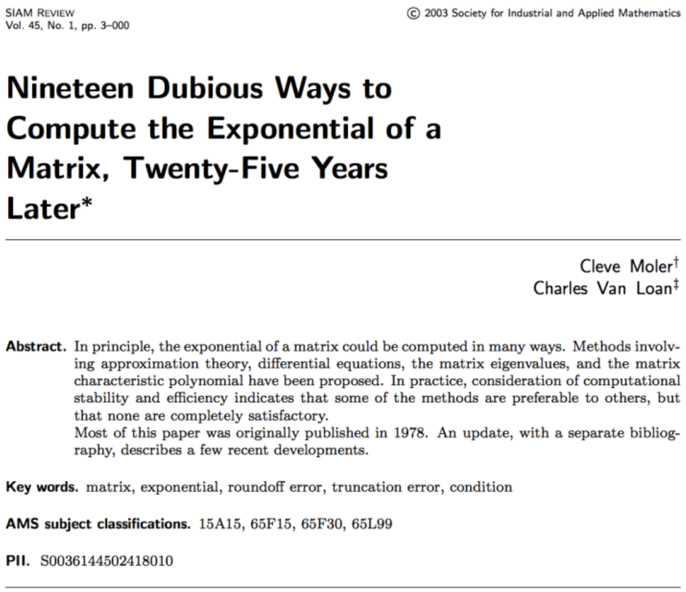
$$\dot{x}(t) = Ax(t) + Bu(t)$$

Matrix exponential:

$$e^{At} = 1 + At + \frac{1}{2!}(At)^2 + \ \ldots = \sum_{0}^{\infty} \frac{1}{k!}(At)^k$$

Theorem. $\Phi(t, t_0) = e^{A(t-t_0)}$, that is

$$x(t, t_0, x_0, u) = x_0 \mathrm{e}^{A(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} Bu(\tau) d\tau$$



SIAM Review
Vol. 45, No. 1, pp. 3–000
© 2003 Society for Industrial and Applied Mathematics

Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*

Cleve Moler[†]
Charles Van Loan[‡]

Abstract. In principle, the exponential of a matrix could be computed in many ways. Methods involving approximation theory, differential equations, the matrix eigenvalues, and the matrix characteristic polynomial have been proposed. In practice, consideration of computational stability and efficiency indicates that some of the methods are preferable to others, but that none are completely satisfactory.

Most of this paper was originally published in 1978. An update, with a separate bibliography, describes a few recent developments.

Key words. matrix, exponential, roundoff error, truncation error, condition

AMS subject classifications. 15A15, 65F15, 65F30, 65L99

PII. S0036144502418010

Example: Gradient flow

- Consider the regression problem of fitting a straight line to data using gradient flow. The behavior of this learning algorithm can be modeled and analyzed as a dynamical system
- Given data: $\{(x_i, y_i)\}_{i=1}^{n}$ we want to fit a linear model $\hat{y}_i = \theta x_i$ by finding the ***best*** θ
- Minimizing the squared loss: $L(\theta) = \frac{1}{2}\sum_i (\theta x_i - y_i)^2$
- Gradient of loss $\nabla_\theta L(\theta) = \frac{1}{2}.2\sum_i (\theta x_i - y_i)x_i = \sum_i (\theta x_i - y_i)x_i = \theta \sum_i x_i^2 - \sum x_i y_i = \theta A - B$
- Gradient descent $\theta_{k+1} = \theta_k - \eta \nabla_\theta L(\theta_k)$, η is a parameter called learning rate

$$= \theta_k - \eta(\mathrm{A}\theta_k - B)$$

- Gradient flow is the continuous-time limit

$$\frac{d\theta(t)}{dt} = -\nabla_\theta L(\theta(t))$$

$$= -A\theta + B$$ LTI system

- Solution $\theta(t) = \theta^* + (\theta(0) - \theta^*)e^{-At}$ $\theta^* = \frac{B}{A}$

$f: \mathbb{R}^n \to \mathbb{R}$

Gradient $\nabla_x f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$

Example: $f(x, y) = x^2 + y^2$

$\nabla_x f = (2x, 2y)$

Gradient flow convergence to an optimal model

Consider the regression problem of fitting a straight line to data using gradient flow. The behavior of this learning algorithm can be modeled and analyzed as a dynamical system

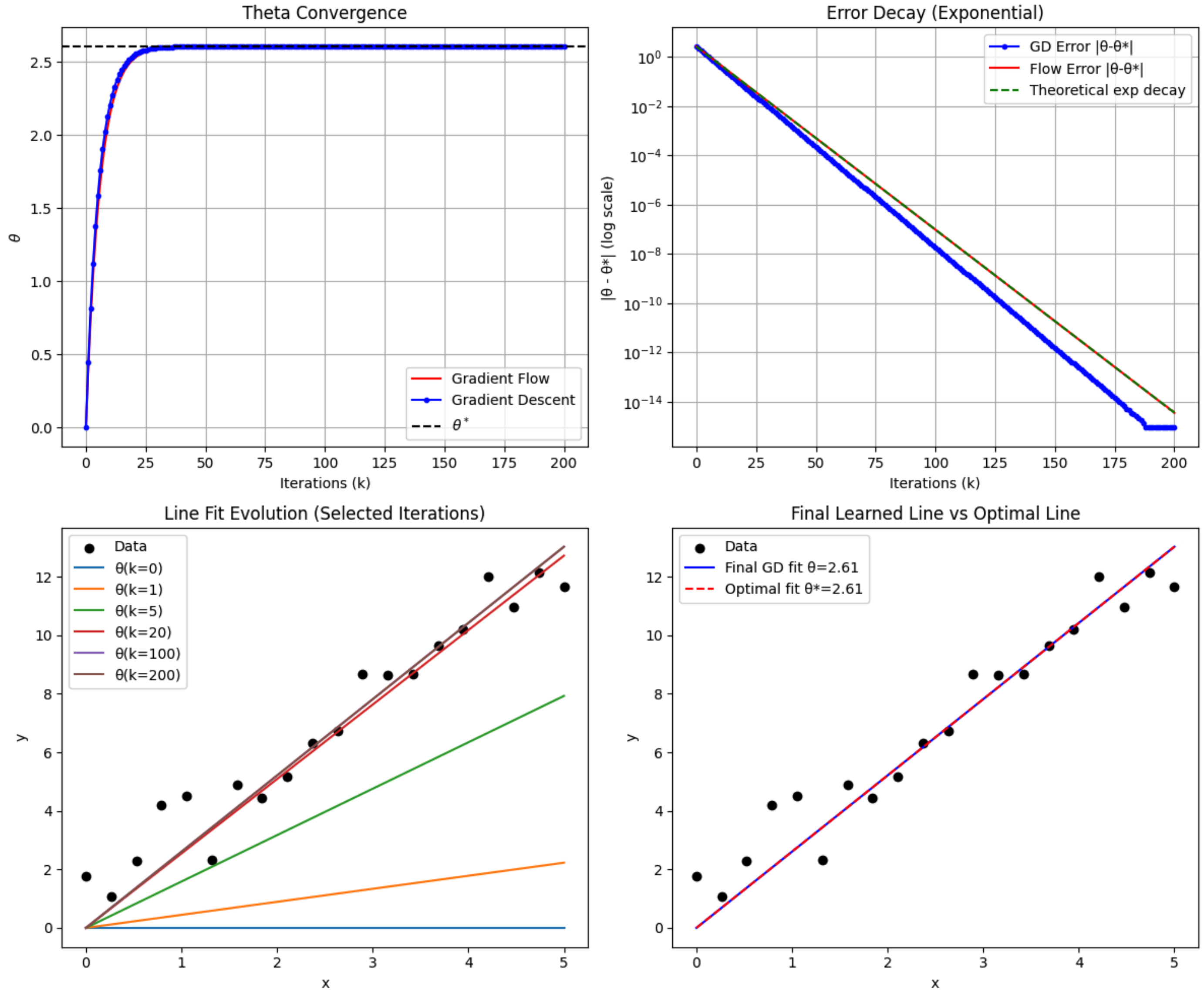
Given data: $\{(x_i, y_i)\}_{i=1}^{n}$ we want to fit a linear model $\hat{y}_i = \theta x_i$ by finding the ***best*** θ

Minimizing the squared loss: $L(\theta) = \frac{1}{2}\sum_i (\theta x_i - y_i)^2$

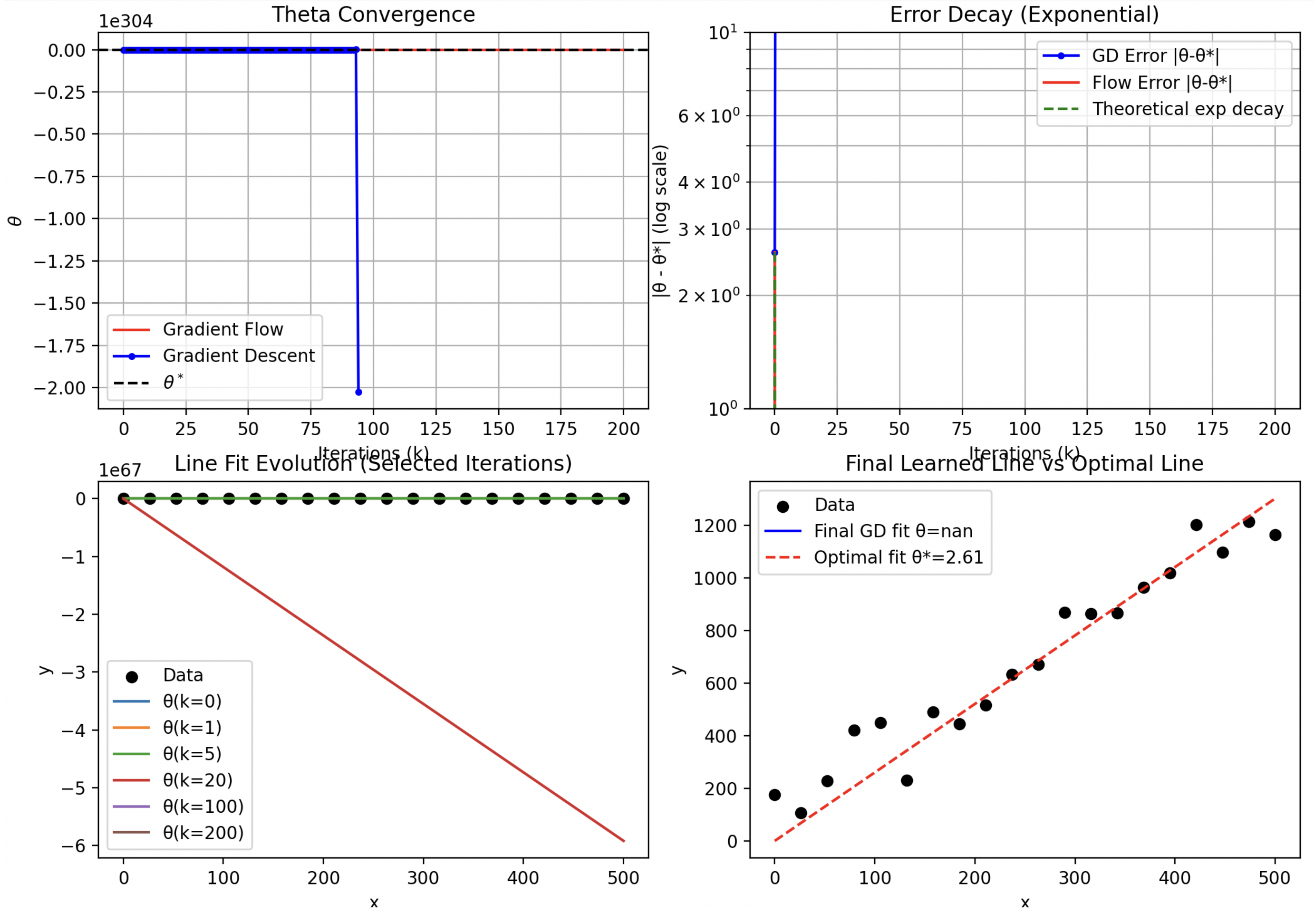
Gradient of loss $\nabla_\theta L(\theta) = \theta A - B$

Gradient descent $\theta_{k+1} = \theta_k - \eta(\mathrm{A}\theta_k - B)$

Gradient flow $\frac{d\theta(t)}{dt} = -A\theta + B$



Grad flow converging but to an undesirable value



Discrete time models / discrete transition systems

- $x(t+1) = f\big(x(t), u(t)\big)$
- $x(t+1) = f(x(t))$ autonomous
- Execution: $x_0, f(x_0), f^2(x_0), \ldots$
- $\boldsymbol{A} = \langle Q, Q_0, T \rangle$
  - $Q = \mathbb{R}^n, Q_0 = \{x_0\}$
  - $T: \mathbb{R}^n \to \mathbb{R}^n; \mathrm{T}(x) = f(x)$
- Deterministic

Discretized or sampled-time model

- $\dot{x}(t) = f\big(x(t), u(t)\big)$
- Assume: $u \in PC(\mathbb{R}, U)\ where\ U \subseteq \mathbb{R}^m$ is a finite set
- $\xi(t, t_0, x_0, u)$
- Fix a sampling period $\delta > 0$
- $\boldsymbol{A_\delta} = \langle Q, Q_0, U, T \rangle$
  - $Q = \mathbb{R}^n, Q_0 = \{x_0\}, Act = U,$
  - $T \subseteq \mathbb{R}^n \times U \times \mathbb{R}^n; (x, u, x') \in \mathrm{T}\ iff\ x' = \xi(\delta, 0, x, u)$