Progress analysis.

How to show that a program always terminates (Halting problem)?

E.g. F(int x)
   While ($x > 1$)
      if x is odd
      else

Does this always terminate?
Unknown! See Collatz Conjecture.
Checked upto $x = 2^{68}$

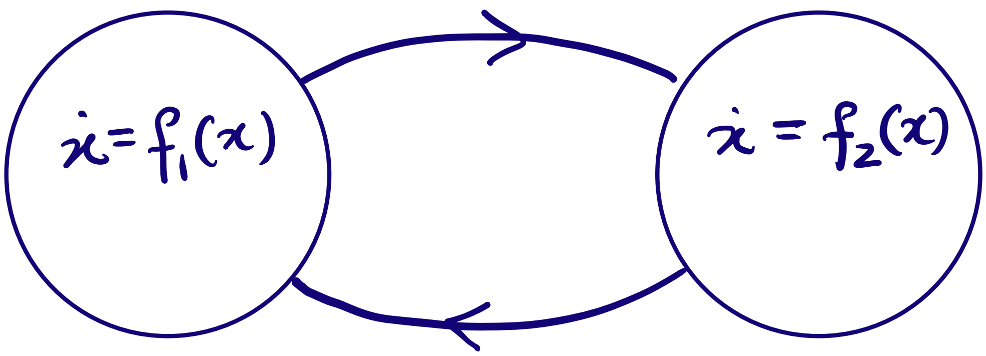
Today. How to check that a given hybrid automaton $\mathcal{A}$ is asymptotically stable.

→ Review : Common Lyapunov Functions
Multiple Lyapunov functions
Stability under slow switchings
→ Dwell time.

Recall our main tool for proving stability of a <u>dynamical system</u> $\dot{x} = f(x)$ $\quad f : \mathbb{R}^n \to \mathbb{R}^n$ was to come up with a positive definite continuous function (Lyapunov function) "Energy"

$V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

What about hybrid systems?



$f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$

We have already seen that even if f_1 and f_2 are individually asymptotically stable, the switchings can make the hybrid system unstable

i.e. V_1 and V_2 Lyapunov functions for f_1 and f_2 is not enough to guarantee stability. What do we need?

Recall. General hybrid Automata

$\mathcal{A} = \langle V, \Theta, A, \mathcal{D}, \mathcal{T} \rangle$

$X \cup \{mode\}$ $\quad$ $type(mode) = \{1, 2, \cdots P\}$

$type(x) = \mathbb{R}$

$A \subseteq \{\langle i, j \rangle\}$ $\quad$ $i, j \in \{1, \cdots, P\}$

A state $\quad v \quad v \lceil mode \quad v \lceil X$

An execution $\alpha = \tau_0 a_1 \tau_1 a_2 \cdots \tau_k$

assume all transitions change the mode i.e.

$v \xrightarrow{a} v' \quad v \lceil mode \neq v' \lceil mode$

Recall

A system is lyapunov stable if

Asymptotically stable if Lyapunov stable and

Common Lyapunov functions (CLF)

Def. A continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$ and $\exists$ a positive definite function $W: \mathbb{R}^n \to \mathbb{R}$ such that for each mode i:

$$\frac{\partial V}{\partial x} f_i(x) < -W(x) \quad \forall x \neq 0$$

Such a V is called a CLF.

Thm. IF there is a Common lyapunov function then the hybrid automaton is globally asymptotically stable.

Proof. Same as proof for dynamical

systems. The function $W(\,)$ gives a lower bound on how slowly the $V(\xi(t))$ value must decrease across all modes j values.

Remark: Finding a CLF involves

(1) Solving many constraints

$$\frac{\partial V}{\partial x} \cdot f_1(x) < a \qquad \frac{\partial V}{\partial x} \cdot f_2(x) < a \; \ldots$$

may not exist.

(2) CLF condition does not rely on discrete transitions at all!

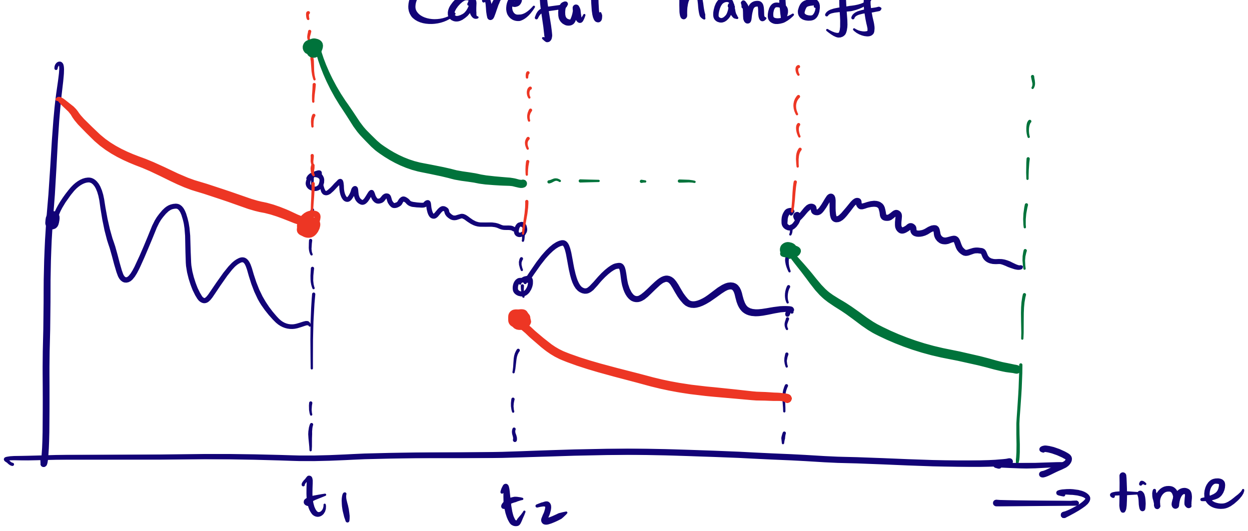
very strong requirement

Multiple Lyapunov Functions (MLF)

Branicky 1998

<u>Idea</u>. different V_i for diff f_i's

Careful "handoff"



<u>Thm</u>. Suppose $\exists\, V_i : \mathbb{R}^n \to \mathbb{R}$ and a positive definite function $W_i : \mathbb{R}^n \to \mathbb{R}$

(i) $\forall i \quad \dfrac{\partial V_i}{\partial x} f_i < 0$

(ii) For any execution $\xi : \mathbb{R} \to \mathbb{R}^n$

$\forall i \qquad V_i(\xi(t_2)) - V_i(\xi(t_1)) \leq -W_i(\xi(t_1))$

for any t_1 and t_2 being the last and first times in mode i. "Handoff" times

<u>Remark</u> "Handoff" condition is not easy to check. Requires reasoning about V_i's across multiple entrance & exit times.

Stability under slow switchings

Hespanha & Morse '99

Idea. Each mode is stable
i.e. has a Lyapunov function V_i
that decays → Energy decays ↓↓

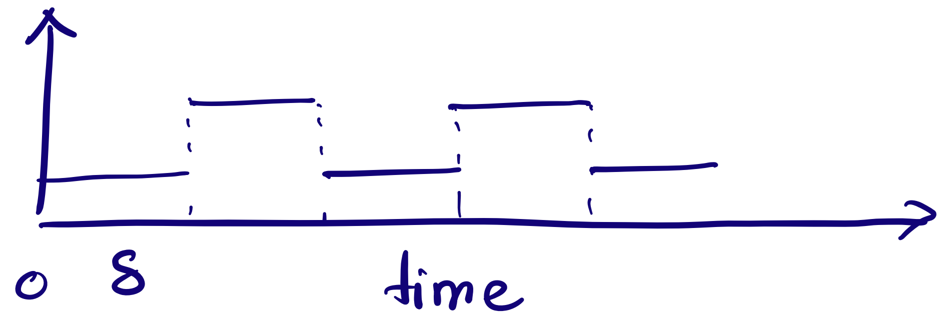
When there is a transition
energy can increase ↑

But, if the increase from transitions
is cancelled by the decay from the
trajectories then overally energy
still decays

So, there should not be "too many"
switches. / transitions

How to define speed of switches ?

Dwell time



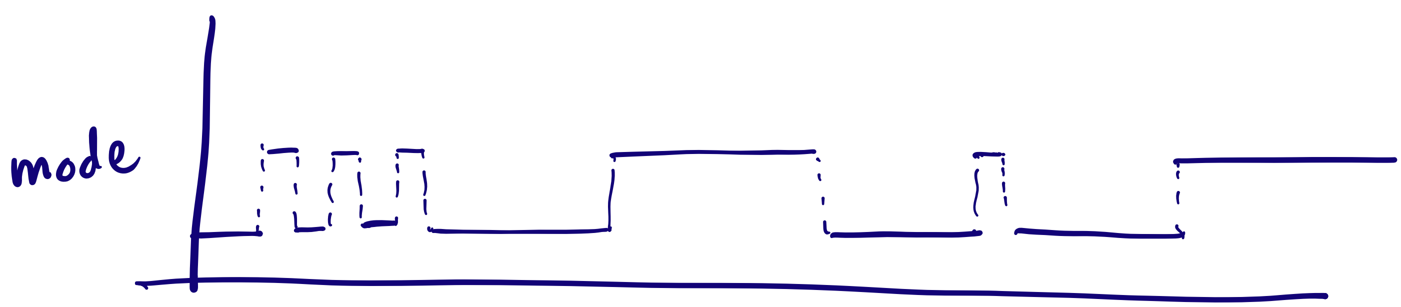
if there is at most 1 switch / transition every δ time then the execution α has a dwell time of δ.

Hybrid automaton $\mathcal{A}$ has dwell time δ if every $\alpha \in Execs_{\mathcal{A}}$ has dwell time δ.

→ is this an invariant?

Average Dwell time (ADT)

At most 1 switch per δ time + a constant No number of extra switches.



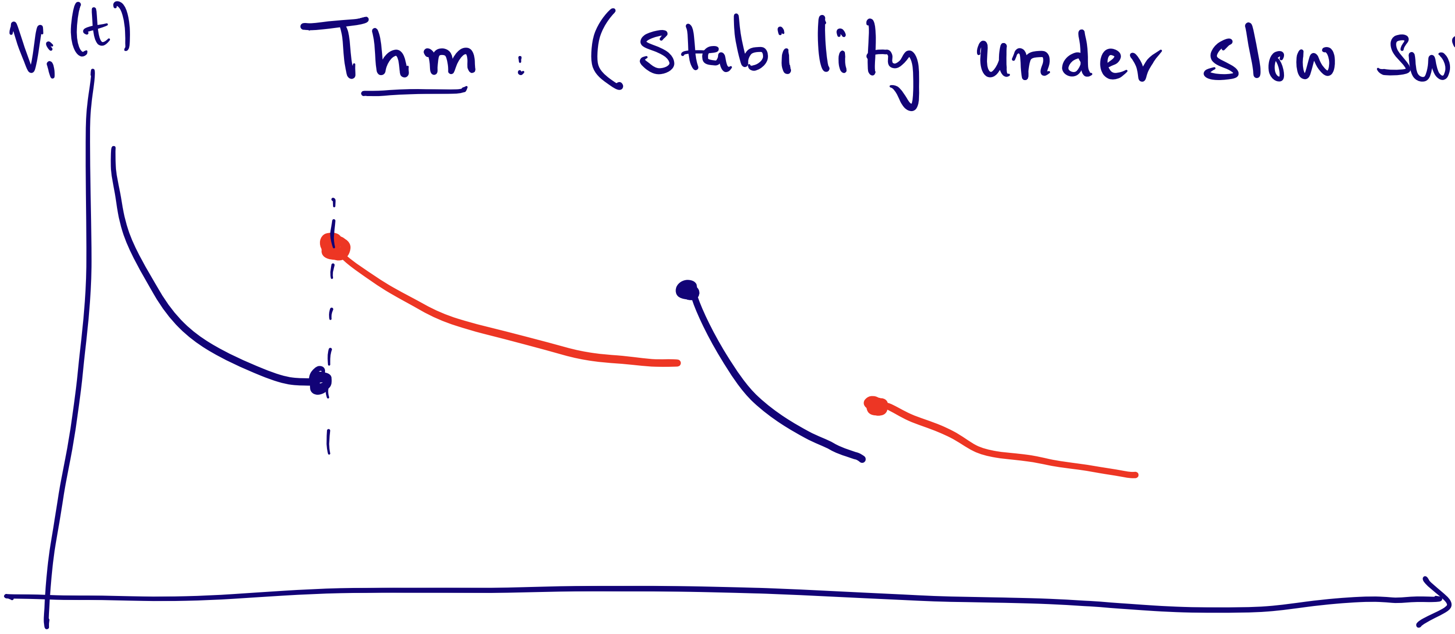
Def. An automaton $\mathcal{A}$ has ADT $\tau_a > 0$ if $\exists N_0 > 0$ $\forall \alpha \in Execs_{\mathcal{A}}$

$N(\alpha)$: # switches / transitions in α
duration(α) : time duration of α.

Q. Is this an invariant?
Can you verify this?

How is this related to zeno?

Thm : (Stability under slow switches)



(1) $\exists \lambda_0 > 0 \ \forall i \quad \dot{V}_i = \frac{\partial V_i}{\partial x} \cdot f_i(x) \le -2\lambda_0 V_i(x)$

an exponentially decaying Lyapunov function for each mode

(2) $\exists \mu$ s.t any transition $v \xrightarrow{a} v'$

$$V_{v'.mode}(v'[x]) \le \mu \ V_{v.mode}(v[x])$$

(3) $\mathcal{A}$ has ADT $> \log \mu / 2\lambda_0$

Then $\mathcal{A}$ is globally asymptotically stable.

Proof. Fix any execution of $\mathcal{A}$ α

Let us look at the prefix of α of duration $T > 0$. Call this prefix α_T

$$\alpha_T = \tau_0 a_1 \tau_1 a_2 \cdots\cdots a_{N(\alpha_T)} \tau_{N(\alpha_T)}$$

The corresponding transition times
$t_1 \; t_2 \; \cdots \; t_{N(\alpha_T)}$

We define a function

$W(t) =$

$W(t)$ is piecewise differentiable
i.e differentiable everywhere except t_i's

$$\frac{dW(t)}{dt} =$$

≤ 0 $\quad$ $W(t)$ is decreasing between t_i and t_{i+1} for each i.

That is, $W(t_{i+1}^-) \leq W(t_i)$

By (ii) $W(t_{i+1}) \leq \mu W(t_{i+1}^-) \leq \mu W(t_i)$

Iterating over $N(\alpha_T)$ transitions

Expanding the definition of W

Summary.

- Progress / Stability proofs rely on Lyapunov / Ranking functions

- Individual Lyapunov functions for the different modes can be "pasted" together to construct stability arguments

  → Multiple Lyapunov functions

  → Stability under slow switches

→ How to prove stability when the individual modes are all <u>not</u> stable?