The next few lectures

Dynamical system models

What is a solution

Lipschitz Continuity

Linear systems

Solution of linear systems

properties of dynamical systems

- Equilibria
- Invariants
- Stability
- Convergence

Verifying Dynamical Systems

Lyapunov methods

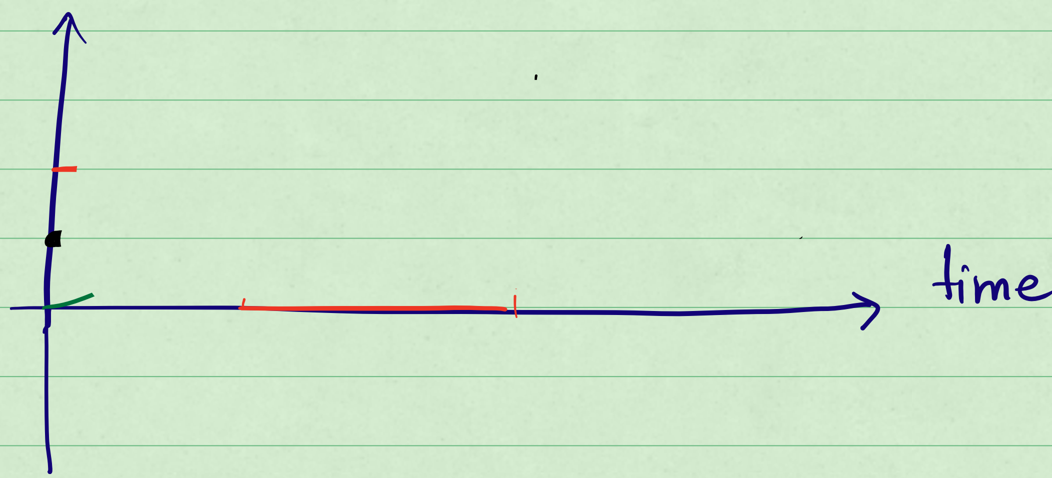
Reachability

Hybrid Systems = Dynamics + Automata

Modeling physical processes

Example model of a vehicle

$$\frac{dx(t)}{dt} = v(t) \qquad \frac{dv}{dt} = a(t)$$



General language for specifying physical laws : Differential Equations

$$\frac{dx(t)}{dt} = f\,(x(t), u(t), t) \qquad \text{— (1)}$$

With no input and if f is not time dependent time invariant then

$$\frac{dx(t)}{dt} = f(x(t)) \qquad \dot{x} = f(x)$$

Used for modeling

Vehicles, weather, Circuits, planetary motion
Biological processes, neurons to populations
Medical devices

Applications:

Prediction
Verification
Controller Design

Examples

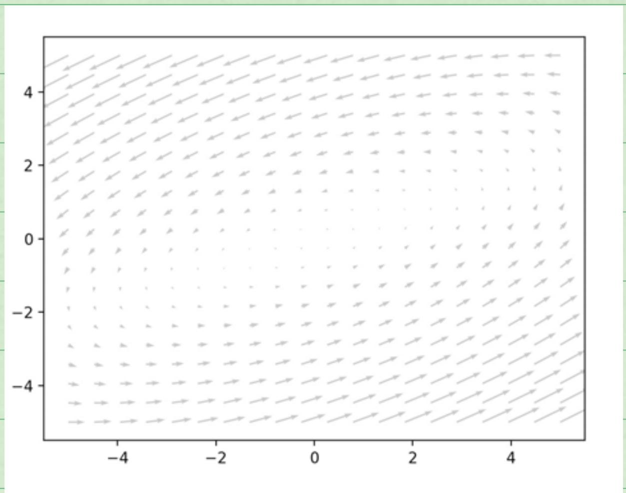
Ex 1

x: national income
y: rate of consumer spend
g: rate of govt expenditur

$$\dot{x} = x - \alpha y$$
$$\dot{y} = \beta(x - y - g)$$

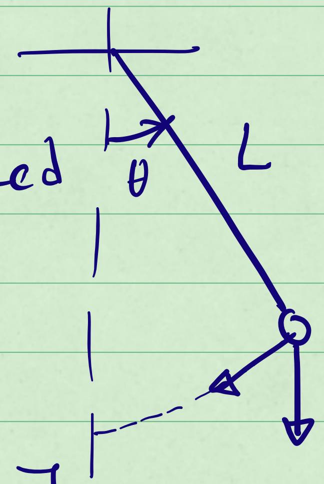
$$g = g_0 + kx$$

$$\dot{y} = \beta(x - y - g_0 - kx)$$
$$= \beta[(1-k)x - y - g_0]$$



Ex 2

Pendulum mass m, length ℓ



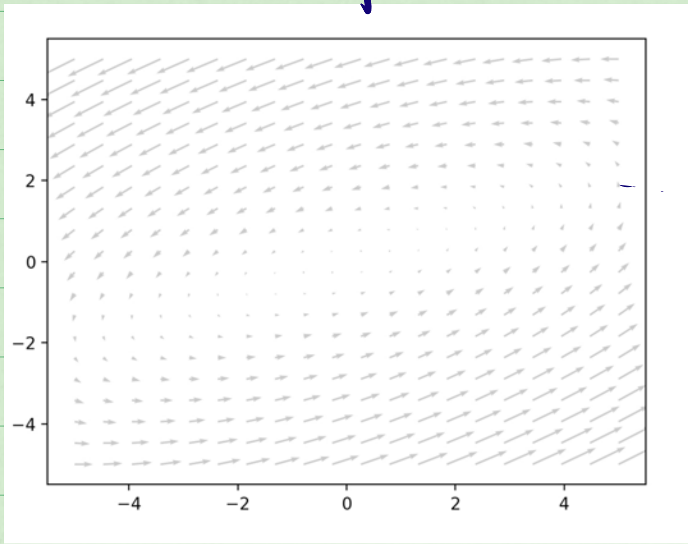
$x_1 = \theta \quad x_2 = \dot{\theta}$ angular speed

$$\frac{dx_2}{dt} = -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - k/m\, x_2 \end{bmatrix}$$

Discrete time

$$f(x) : \mathbb{R}^2 \to \mathbb{R}^2$$



What is a solution of (1) ?

Given an initial state $x_0 \in \mathbb{R}^n$
and an input signal $u : \mathbb{R} \to \mathbb{R}^m$ a
function $\xi : \mathbb{R} \to \mathbb{R}^n$ is a <u>solution</u> of (1)
iff (1)

(2)

Problems with this definition?

- Assumes $\xi(t)$ is differentiable at all $t \in \mathbb{R}$.

- If $u(t)$ is discontinuous then $\xi(t)$ cannot be differentriable everywhere

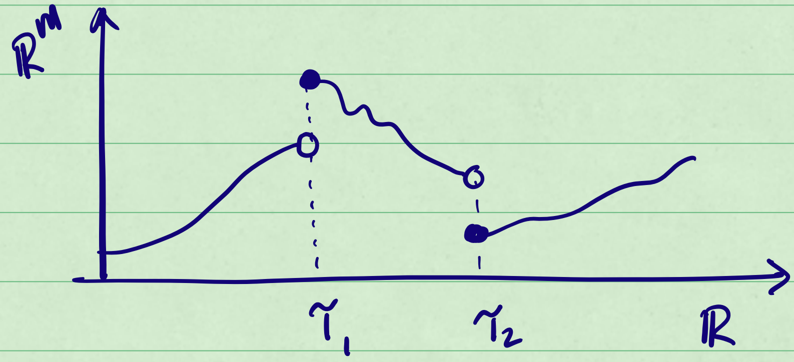
<u>Def.</u> $u : \mathbb{R} \to \mathbb{R}^m$ is piece-wise continuous with a set of discontinuity points $D \subseteq \mathbb{R}$ if

(1) $\forall \tau \in D \quad \lim_{t \to \tau^+} u(t) < \infty \quad \lim_{t \to \tau^-} u(t) < \infty$

Limits exist

(2) Continuous from right $\lim_{t \to \tau^+} u(t) = u(t)$

(3) $\forall t_0 < t_1 \in \mathbb{R} \quad [t_0, t_1] \cap D$ is finite

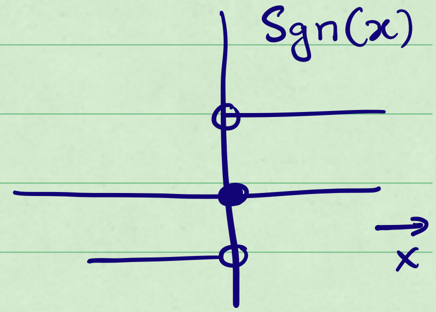


Modified definition of solution (2) $\forall t \in \mathbb{R} \setminus D$.

Do solutions exist?

<u>Example</u> $\dot{x} = f(x)$

$$f(x) = -\,\mathrm{sgn}(x)$$

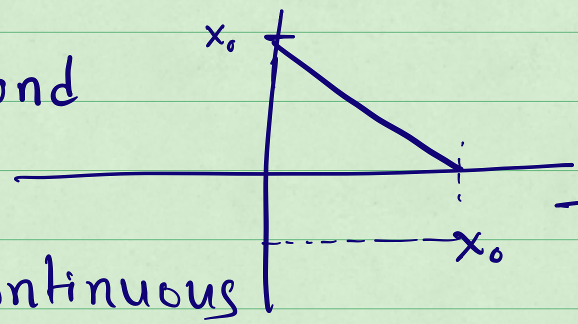


Solution: $\xi : \mathbb{R} \to \mathbb{R}$

$$\xi(t) = x_0 - t \quad , \quad x_0 > 0$$

check $\dfrac{d\xi(t)}{dt} = -1 = -\,\mathrm{sgn}(\xi(t))$

Not defined beyond $t \geqslant x_0$



<u>problem</u> f is discontinuous

<u>Example</u>. $\dot{x} = x^2 \qquad x_0 \in \mathbb{R}$

Solution. $\xi(t) = \dfrac{x_0}{1 - t x_0}$

<u>Check</u> $\dfrac{d\xi(t)}{dt} = \dfrac{(-1)\,x_0 \cdot (-x_0)}{(1 - t x_0)^2} = (\xi(t))^2$

But as $t \to 1/x_0 \quad \xi(t) \to \infty$

Problem $f(x)$ grows too fast.

We require f to be Lipschitz continuous.

$\exists L > 0$ such that

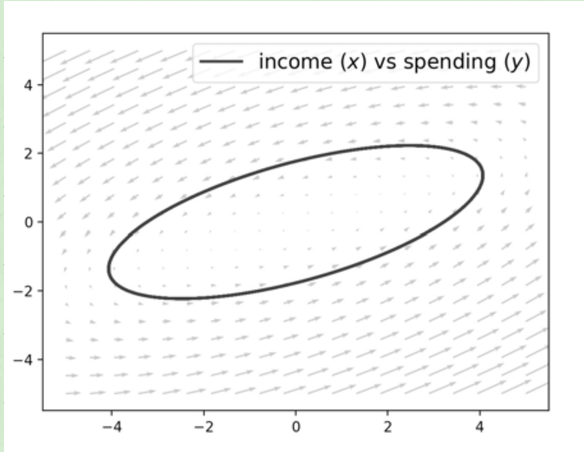
$$\forall x, x' \quad |f(x) - f(x')| \leq L\,|x - x'|$$

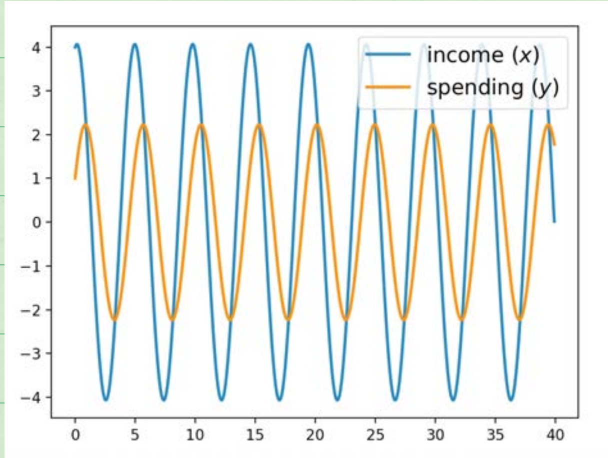
Examples.

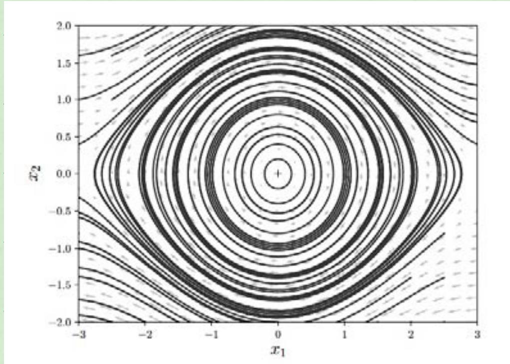
Non Examples

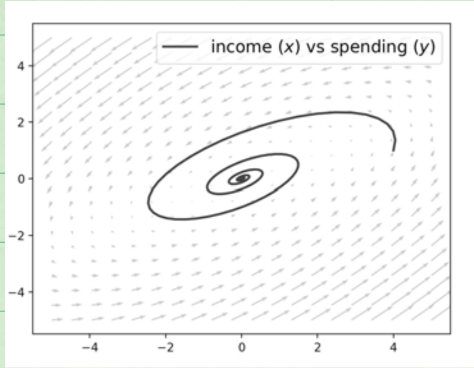
Thm

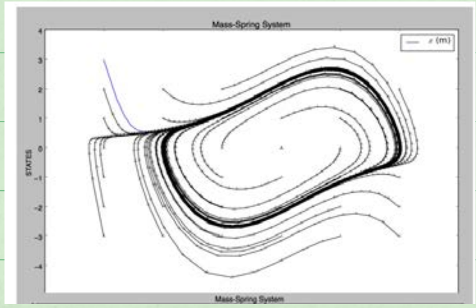
If $f(x,u)$ is Lipschitz continuous in the first argument then (1) has unique solutions.

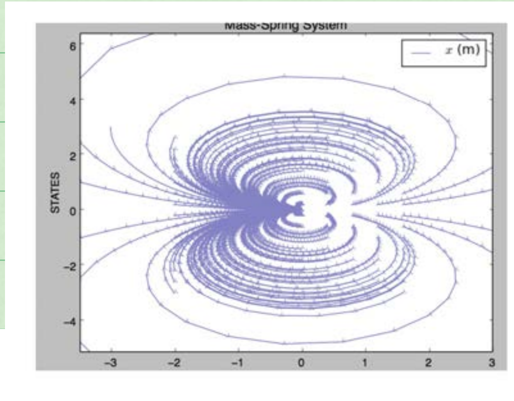


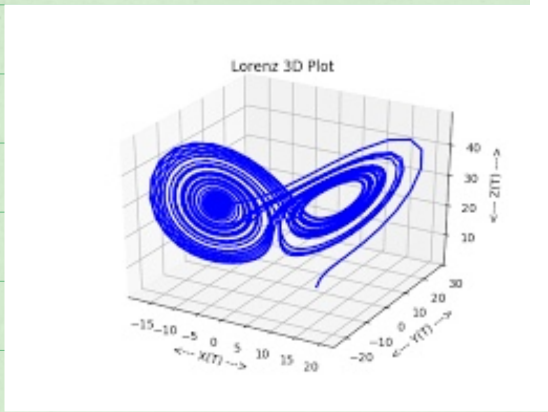












Linear Dynamical Systems

$$\frac{dx(t)}{dt} = f(x(t), u(t))$$

$u(t)$ is continuous $\forall t \in \mathbb{R}/D$

f is Linear function

Therefore Lipschitz in $x(t)$

$$\frac{dx}{dt} = A(t)\,x(t) + B(t)\,u(t) \tag{2}$$

$A(t)$, $B(t)$ are matrices of appropriate dimensions

<u>Theorem</u> Let $\xi(t, x_0, u)$ be the solution of (2) with points of discontinuity D.

(1) $\forall x_0, U$ $\forall t \in \mathbb{R}/D$ $\xi(t, x_0, u)$ is continuous and differentiable w.r.t t.

(2) $\forall t, u$ $\forall x_0$ $\xi(t, x_0, u)$ is continuous w.r.t. x_0

(3) $\forall t$ $x_{01}\ x_{02}$ $u_{01}\ u_{02}$ $a_1\, a_2 \in \mathbb{R}$

$$\xi(t, a_1 x_{01} + a_2 x_{02}, a_1 u_{01} + a_2 u_{02}) =$$

$$a_1 \xi(t, x_{01}, u_{01}) + a_2 \xi(t, x_{02}, u_{02})$$

(4) $\xi(t, x_0, U) = \xi(t, x_0, \vec{0}) + \xi(t, 0, u)$

Theorem Linear time invariant (LTI) System $A(t) = A \quad \forall t$

Solution

$$\xi(t, x_0, u) = x_0 e^{A(t-t_0)} + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau$$

IF there are no inputs

IF the control input $u(t) = -Kx(t)$

then $\dot{x} =$

Requirements for Dynamical Systems

We focus on closed systems

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

What are the equilibria or Stationary points?

$x^* \in \mathbb{R}^n$ is an <u>equilibrium</u> if

examples.

Pendulum. $f : \begin{bmatrix} -g/l \sin(x_1) - \frac{k}{m} x_2 \\ x_2 \end{bmatrix}$

Economy : $\dot{x} = x - \alpha y = 0 \qquad x = \alpha y$

$\dot{y} = \beta(x - y - g_0 - kx) = 0$

$$\beta(\alpha - 1 - k\alpha)x = g_0 \beta$$

$$x^* = \frac{g_0}{\alpha - 1 - k\alpha}$$

W.l.o.g we assume that $\vec{0}$ is an equilibrium point

Stability : How does the system behave near an equilibrium?

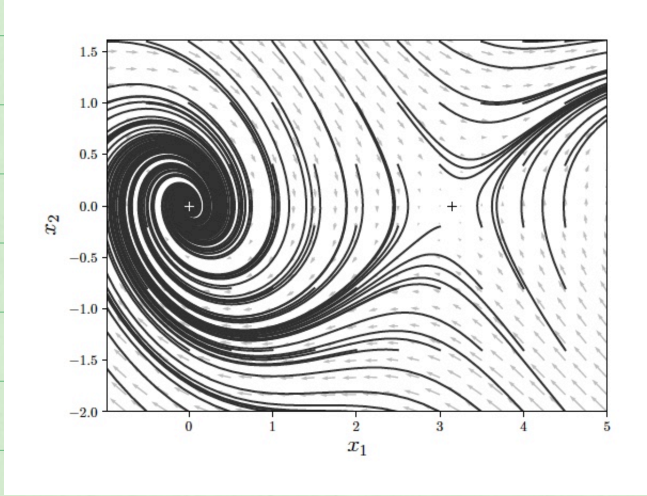
Does it stay bounded, Converge diverge? Are there invariants?

Def

A dynamical system is stable (Lyapunov stable) at the origin if

Otherwise (1) is said to be Unstable.

How is Lyapunov stability related to invariance?



Asymptotic stability : A dynamical system is asymptotically stable if

(1)

(2)

IF (2) holds for all δ_2 then
Globally Asymptotically Stable (GAS).