How to prove stability of a dynamical system

$\dot{x} = f(x) \quad x \in \mathbb{R}^n \; f: \mathbb{R}^n \to \mathbb{R}^n$?

A function $f: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is positive definite if

$f(x) = 0$ iff $x = 0$

<u>Thm</u> Suppose there exists a positive definite continuous function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$.

(i) if $\dot{V} \leq 0$ then the dynamical system is Lyapunov stable

(ii) if $\forall x \neq 0 \quad \dot{V} < 0$ then the system is globally asymptotically stable

<u>Remarks</u> (1) sufficient condition

(2) Compare with theorem 7.1 (inductive invariance)

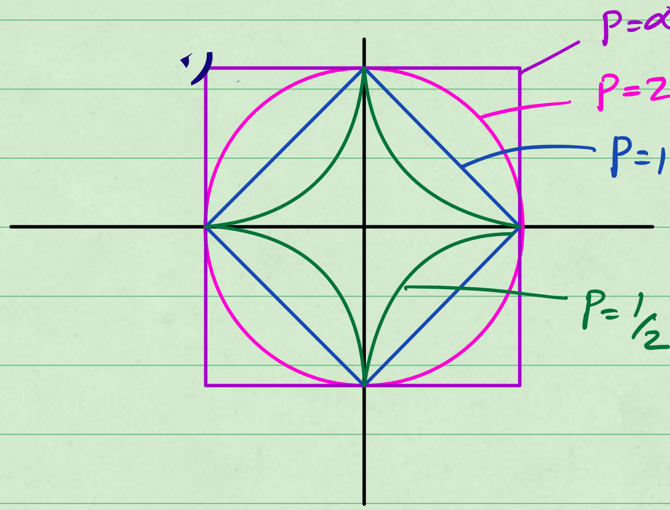
What is $\dot{V}$?

Applying the chain rule

Eg $V = 6x^2 + 4$

Notations : Balls

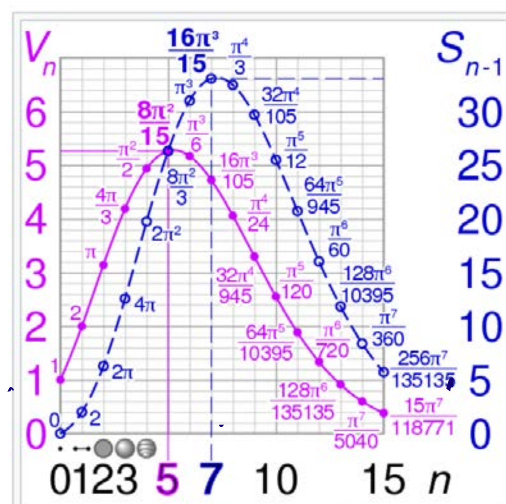
A ball of radius $r > 0$ at $x \in \mathbb{R}^n$



interesting (but for now irrelevant) fact

for $r = 1$ (unit radius) what is the volume of an n-dimensional sphere of unit radius

$B(0,1)$



Graphs of volumes (V) and surface areas (S) of n-balls of radius 1. In the SVG file, hover over a point to highlight it and its value.

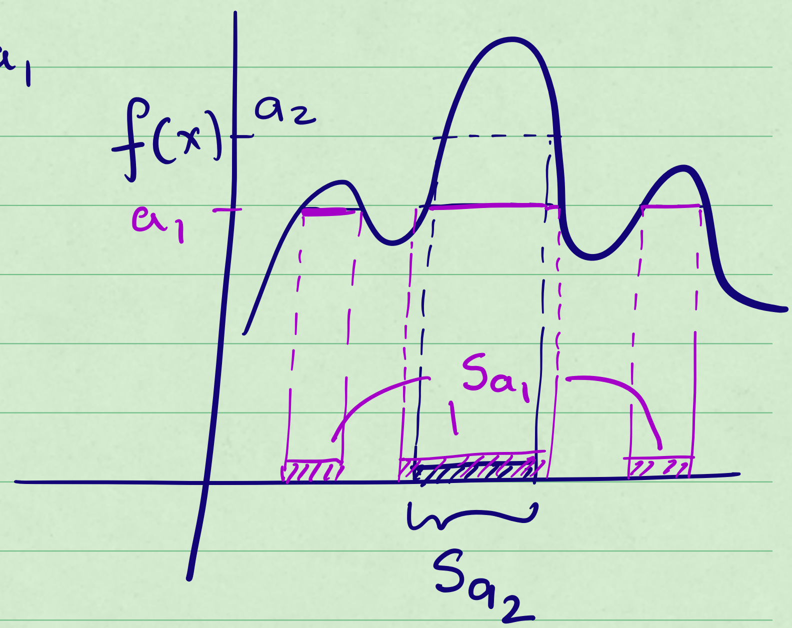
Level sets and Sublevel-Sets

for any function $f: \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}$, the <u>a-level set</u> of f is the set $L_a = \{x \in \mathbb{R}^n \mid f(x) = a\}$

a-Sublevel set $S_a = \{x \in \mathbb{R}^n \mid f(x) \leq a\}$

Obviously $a_1 \leq a_2 \Rightarrow$

$S_{a_2} \subseteq S_{a_1}$



$B(0,r) = S_r$
for $|x|$

Proof.

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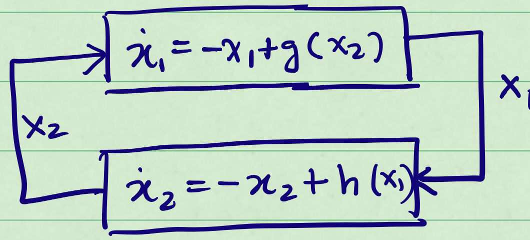
Example

$\dot{x}_1 = -x_1 + g(x_2)$

$\dot{x}_2 = -x_2 + h(x_1)$



given $|g(x_2)| \leq \frac{|x_2|}{2}$

$|h(x_1)| \leq \frac{|x_1|}{2}$

What can we say about stability of the closed system?

Define $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$

$\dot{V} =$

Therefore the system is globally asymptotically stable [we did not have to calculate solutions. In fact the system is incompletely defined]

How to find Lyapunov functions?

Guess a template (e.g. quadratic)

$$V(x) = a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 + a_4 x_1 + a_5 x_2 + a_6$$

Then solve for parameters $a_1 \ldots a_6$ with constraints

$$\frac{\partial V}{\partial x} \cdot f(x) \leq 0$$

Can often be solved using Convex optimization

For linear systems $\dot{x} = Ax \quad x \in \mathbb{R}^n$

<u>Thm</u> there is always a quadratic Lyapunov function if the system is stable.

Quadratic function $\quad x^T x$

$V(x) :=$

$\dot{V}(x) =$

$=$

$=$

$=$

that is Fix Q to be positive definite and symmetric matrix $\mathbb{R}^{n \times n}$

e.g. $\quad Q = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \cdots & & & \end{bmatrix} = I$

Solve for P in $A^T P + PA = -Q$

Lyapunov equation.

[illegible]