The next few lectures

Dynamical system models
What is a solution
Lipschitz Continuity

Linear systems
Solution of linear systems

properties of dynamical systems
- Equilibria
- Invariants
- Stability
- Convergence

Verifying Dynamical Systems

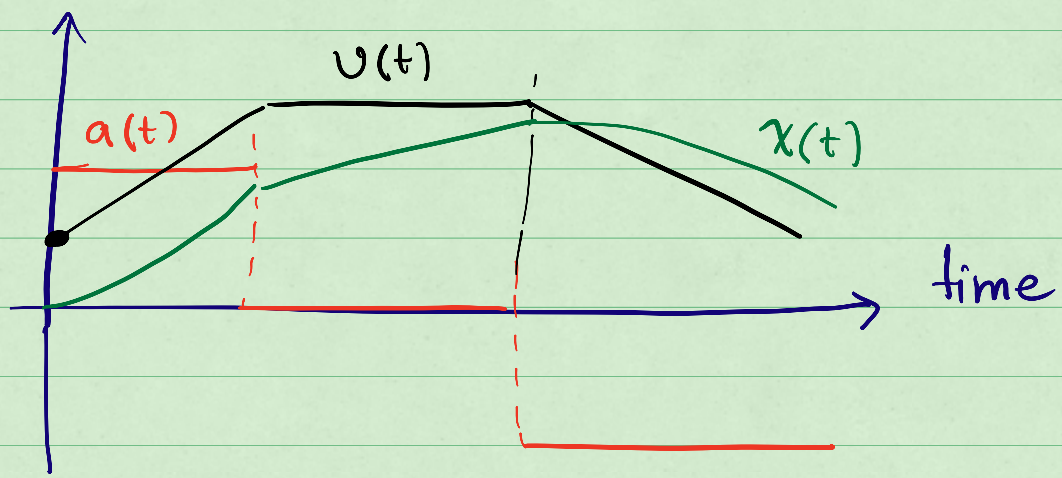
Lyapunov methods

Reachability

Modeling physical processes

Example model of a vehicle

$$\frac{dx(t)}{dt} = v(t) \qquad \frac{dv}{dt} = a(t)$$



General language for specifying physical laws : Differential Equations

$$\frac{dx(t)}{dt} = f\,(x(t), u(t), t) \qquad \text{--- (1)}$$

$t \in \mathbb{R}$

$x(t) \in \mathbb{R}^n$ state

$u(t) \in \mathbb{R}^m$ input /Control

$f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$

With no input and if f is not time dependent time invariant then

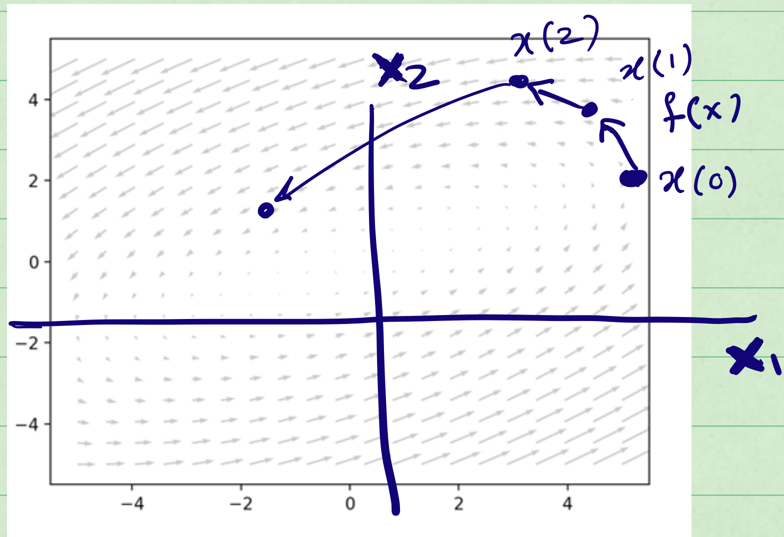
$$\frac{dx(t)}{dt} = f(x(t)) \qquad \dot{x} = f(x)$$

Used for modeling

Vehicles, weather, Circuits, Planetary motion
Biological processes, neurons to populations
Medical devices

Discrete time analog

$$x(t+1) = f(x(t))$$



What is a solution of (1)?

Given an initial state $x_0 \in \mathbb{R}^n$
and an input signal $u : \mathbb{R} \to \mathbb{R}^m$ a
function $\xi : \mathbb{R} \to \mathbb{R}^n$ is a <u>solution</u> of (1)
iff $\xi(0) = x_0$ and

$$\forall t \quad \frac{d}{dt}\xi(t) = f(\xi(t), u(t), t)$$

Problems with this definition?

- Assumes $\xi(t)$ is differentiable at all $t \in \mathbb{R}$.

- If $u(t)$ is discontinuous then $\xi(t)$ cannot be differentiable everywhere

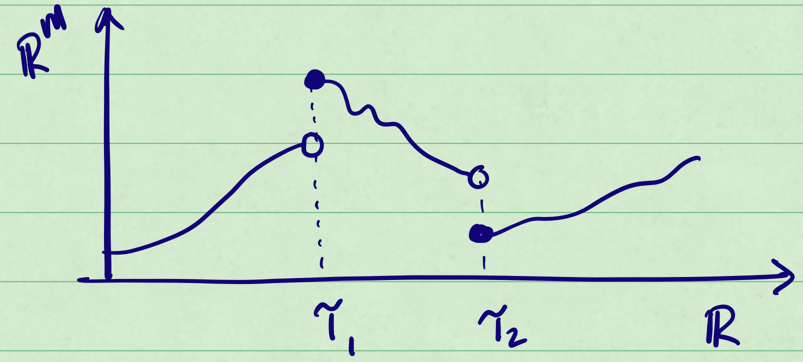
Def. $u : \mathbb{R} \to \mathbb{R}^m$ is piece-wise continuous with a set of discontinuity points $D \subseteq \mathbb{R}$ if

(1) $\forall \tau \in D \quad \lim_{t \to \tau^+} u(t) < \infty \quad \lim_{t \to \tau^-} u(t) < \infty$

Limits exist

(2) Continuous from right $\lim_{t \to \tau^+} u(t) = u(t)$

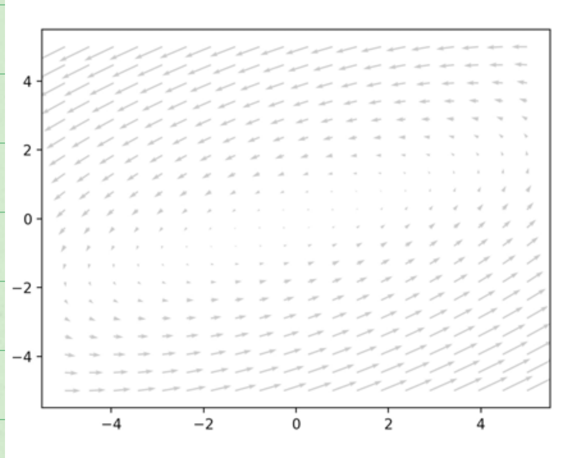
(3) $\forall t_0 < t_1 \in \mathbb{R} \quad [t_0, t_1] \cap D$ is finite



Modified definition of solution (2) $\forall t \in \mathbb{R} \setminus D$.

Examples / Code

Vehicle



Economy model.

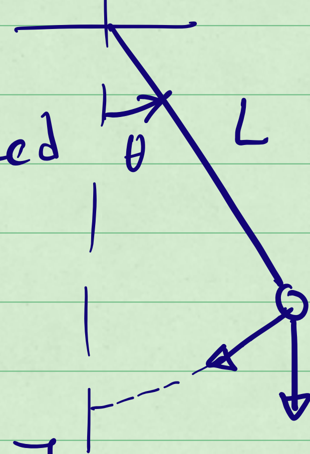
x: national income
y: rate of consumer spend
g: rate of govt expenditure

$$\dot{x} = x - \alpha y$$
$$\dot{y} = \beta(x - y - g)$$

$$g = g_0 + kx$$

$$\dot{y} = \beta(x - y - g_0 - kx)$$
$$= \beta[(1-k)x - y - g_0]$$

<u>Pendulum</u> mass m, length ℓ

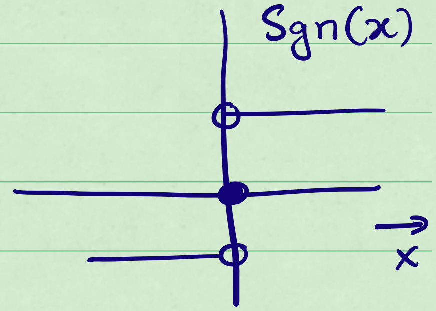


$x_1 = \theta \quad x_2 = \dot{\theta}$ angular speed

$$\frac{dx_2}{dt} = -\frac{g}{\ell}\sin x_1 - \frac{k}{m} x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - k/m\, x_2 \end{bmatrix}$$

Do solutions exist?



Example $\dot{x} = f(x)$

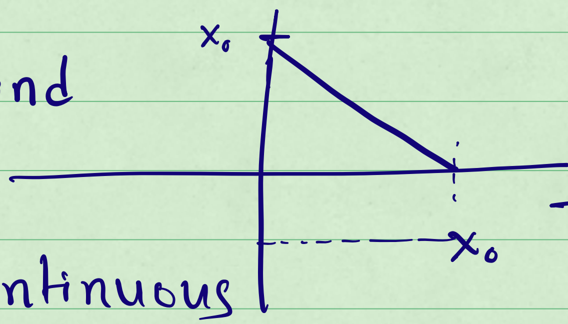
$f(x) = -\operatorname{sgn}(x)$

Solution: $\xi : \mathbb{R} \to \mathbb{R}$

$\xi(t) = x_0 - t \quad , \; x_0 > 0$

check $\dfrac{d\xi(t)}{dt} = -1 = -\operatorname{sgn}(\xi(t))$

Not defined beyond $t \geqslant x_0$



Problem f is discontinuous

Example. $\dot{x} = x^2 \qquad x_0 \in \mathbb{R}$

Solution. $\xi(t) = \dfrac{x_0}{1 - t x_0}$

Check $\dfrac{d\xi(t)}{dt} = \dfrac{(-1)\, x_0 \, . \, (-x_0)}{(1 - t x_0)^2} = (\xi(t))^2$

But as $t \to 1/x_0 \quad \xi(t) \to \infty$

Problem $f(x)$ grows too fast.

We require f to be Lipschitz continuous.

$\exists L > 0$ such that

$$\forall x, x' \quad |f(x) - f(x')| \leq L |x - x'|$$

<u>Examples.</u> $f(x) = ax + b$, $\sin(x)$

<u>Non Examples</u> e^x, x^2, $\sqrt{x}$

<u>Thm</u>
If $f(x,u)$ is Lipschitz continuous in the first argument then (1) has unique solutions.

Recall a solution from a given initial state x_0 is a function of time

$\xi : \mathbb{R} \to \mathbb{R}^n$ $\quad \xi(t)$ is the state at time t, starting from x_0.

If we want to make the debendence explicit then $\xi(x_0, t)$

$\xi : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$

From a set of initial states $\Theta \subseteq \mathbb{R}^n$ the set of <u>Reachable states</u> upto $T > 0$

$$\mathrm{Reach}(\Theta, T) \triangleq \{x \in \mathbb{R}^n \mid \exists\, x_0 \in \Theta, t \leq T, \xi(x_0, t) = x\}$$

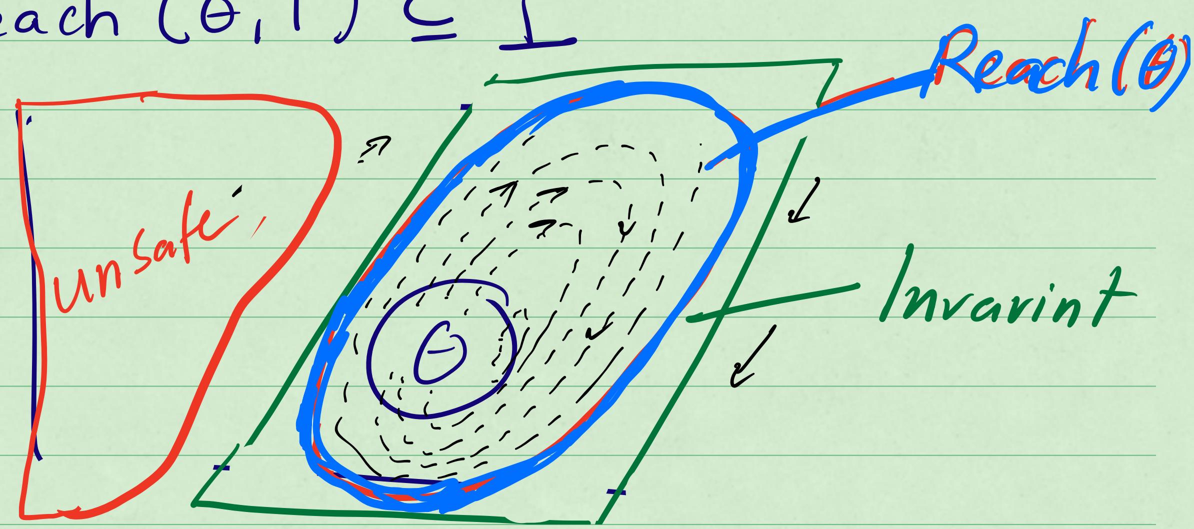
$$\mathrm{Reach}(\Theta) = \mathrm{lt}_{T \to \infty} \mathrm{Reach}(\Theta, T)$$

Unbounded time reach set / space

Related concept Controllable / space

An invariant of (1) is a set of states $I \subseteq \mathbb{R}^n$ such that

$$\mathrm{Reach}(\Theta, T) \subseteq I$$



If the system has input then given x_0, $u : \mathbb{R} \to \mathbb{R}^m$ the solution is $\xi(x_0, u, t)$

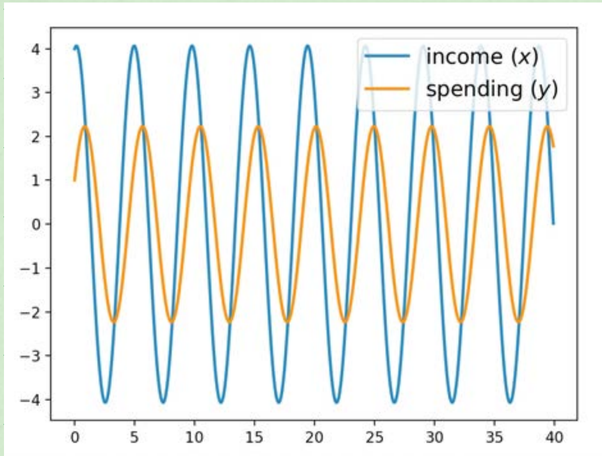
Linear Dynamical Systems

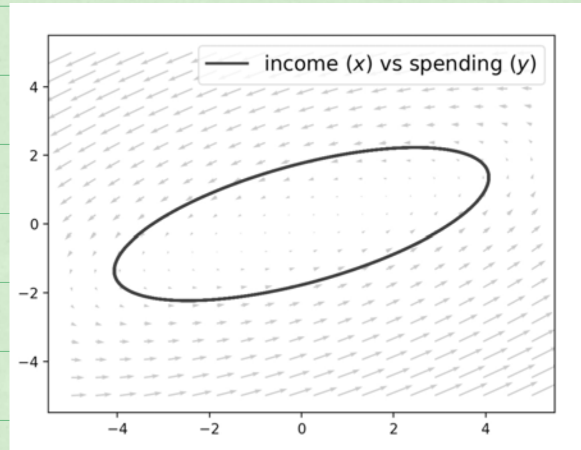
$$\frac{dx(t)}{dt} = f(x(t), u(t))$$

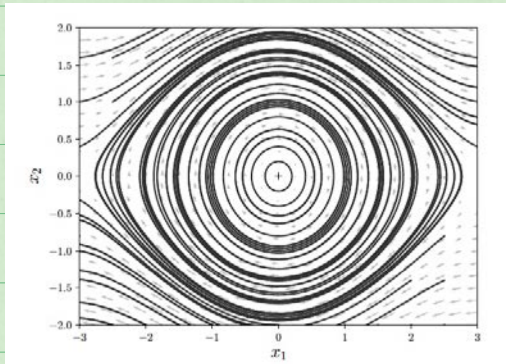
$u(t)$ is continuous $\forall t \in \mathbb{R}/D$

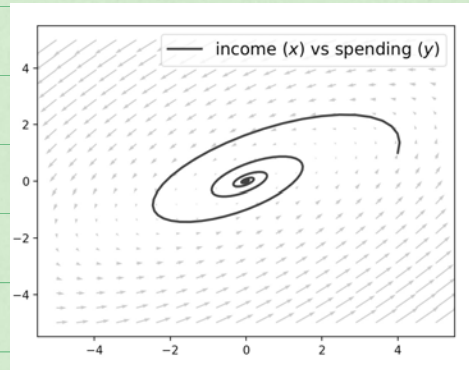
f is Linear function

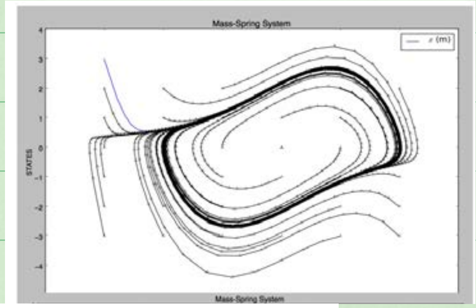
Therefore Lipschitz in $x(t)$

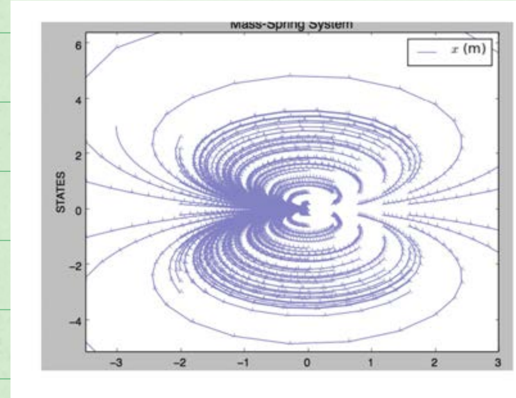

income (x)
spending (y)

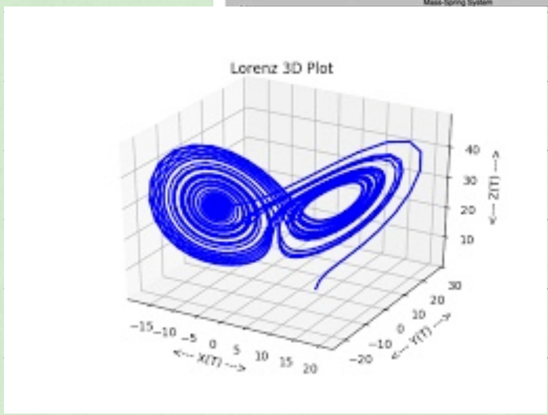

income (x) vs spending (y)


x_2
x_1


income (x) vs spending (y)


Mass-Spring System
STATES
Mass-Spring System


Mass-Spring System
x (m)
STATES


Lorenz 3D Plot
<--- X(T) --->
<--- Y(T) --->
<--- Z(T) --->

$$\frac{dx}{dt} = A(t)\,x(t) + B(t)\,u(t) \qquad (2)$$

$A(t)$, $B(t)$ are matrices of approprite dimensions

<u>Theorem</u> Let $\xi(t, x_0, u)$ be the solution of (2) with points of discontinuity D.

(1) $\forall x_0, U \;\; \forall t \in \mathbb{R}/D \;\; \xi(t, x_0, u)$ is continuous and differentiable w.r.t t.

(2) $\forall t, u \;\; \forall x_0 \;\; \xi(t, x_0, u)$ is continuous w.r.t. x_0

(3) $\forall t \;\; x_{01} \; x_{02} \;\; u_{01} \; u_{02} \;\; a_1 a_2 \in \mathbb{R}$

$$\xi(t, a_1 x_{01} + a_2 x_{02}, a_1 u_{01} + a_2 u_{02}) = a_1 \xi(t, x_{01}, u_{01}) + a_2 \xi(t, x_{02}, u_{02})$$

(4) $\xi(t, x_0, u) = \xi(t, x_0, \vec{0}) + \xi(t, 0, u)$

Theorem Linear time invariant (LTI) System $A(t) = A \quad \forall t$

$$\dot{x} = Ax + Bu(t)$$

Solution

$$\xi(t, x_0, u) = x_0 e^{A(t-t_0)} + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau$$

IF there are no inputs

$$\boxed{\dot{x} = Ax \qquad \xi(t, x_0) = x_0 e^{At}}$$

IF the control input $u(t) = -Kx(t)$

then
$$\begin{aligned}
\dot{x} &= Ax(t) + Bu(t) \\
&= Ax(t) - BKx(t) \\
&= (A - BK)\, x(t) \\
&= A'\, x(t)
\end{aligned}$$

Requirements for Dynamical Systems

We focus on closed systems

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

What are the equilibria or Stationary points?

$x^* \in \mathbb{R}^n$ is an <u>equilibrium</u> if $f(x^*) = 0$

examples.

Pendulum. $f : \begin{bmatrix} -g/l \sin(x_1) - \frac{k}{m} x_2 \\ x_2 \end{bmatrix}$

Economy : $\dot{x} = x - \alpha y = 0 \qquad x = \alpha y$
$\dot{y} = \beta(x - y - g_0 - kx) = 0$

$$\beta(\alpha - 1 - k\alpha)x = g_0\beta$$

$$x^* = \frac{g_0}{\alpha - 1 - k\alpha}$$

W.l.o.g we assume that $\vec{0}$ is an equilibrium point

Stability : How does the system behave near an equilibrium?

Does it stay bounded, converge diverge? Are there invariants?

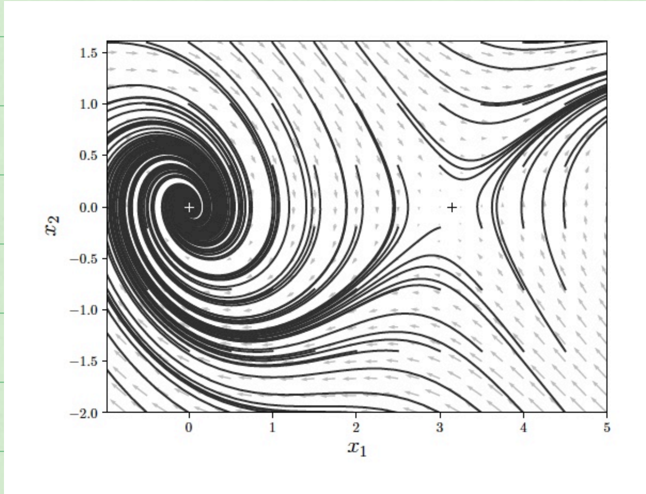
Def

A dynamical system is stable (Lyapunov stable) at the origin if

$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 \quad \text{if } |x_0| \leq \delta_\varepsilon \text{ then}$$
$$\forall t \geq 0 \quad |\xi(x_0, t)| \leq \varepsilon$$

Otherwise (1.) is said to be Unstable.

How is Lyapunov stability related to invariance?

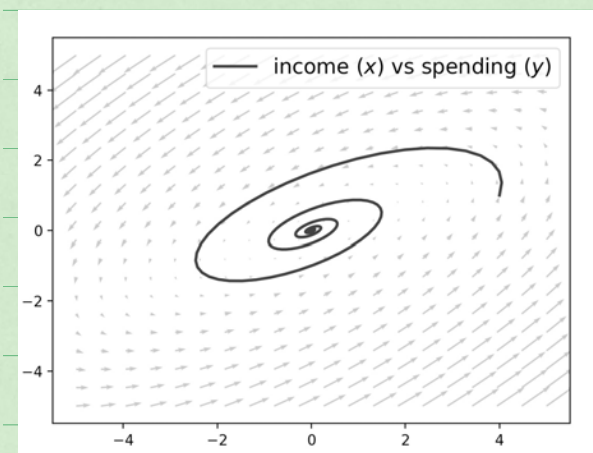


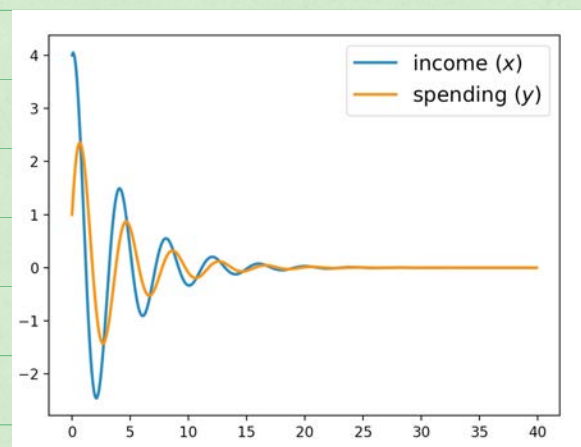
Asymptotic stability : A dynamical system is asymptotically stable if

(1) it is Lyapunov stable and

(2) $\exists\, \delta_2 > 0$ s.t. $|x_0| \leq \delta_2 \Rightarrow$ as $t \to \infty$ $|\underset{\sim}{\xi}(x_0, t)| \longrightarrow \vec{0}$

IF (2) holds for all δ_2 then Globally Asymptotically Stable (GAS).





Exponentially stable if $\exists\, \delta_3, c, \lambda > 0$ such that $|\underset{\sim}{\xi}(t)| \leq c\,|\underset{\sim}{\xi}(0)|\, e^{-\lambda t}$ for any $|\underset{\sim}{\xi}(0)| \leq \delta_3$.